L^2-THEORY FOR THE $\overline{\partial}$-OPERATOR ON COMPACT COMPLEX SPACES

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ABSTRACT. In the 1960s, the L^2-theory for the $\overline{\partial}$-operator has become an important, indispensable part of complex analysis with influence to many other areas of mathematics. But, whereas the theory is very well developed on complex manifolds, it has been an open problem ever since to create an appropriate L^2-theory for the $\overline{\partial}$-operator on singular complex spaces. In the present article, we make some steps towards such a theory for $(0, q)$ and (n, q)-forms on compact complex spaces of pure dimension n with isolated singularities. Some of our methods apply to arbitrary singularities and non-compact situations.

1. INTRODUCTION

In the 1960s, the L^2-theory for the $\overline{\partial}$-operator has become an important, indispensable part of complex analysis through the fundamental work of Hörmander on L^2-estimates and existence theorems for the $\overline{\partial}$-operator (see [H3] and [H4]) and the related work of Andreotti and Vesentini (see [AV]).[1] One should also mention Kohn's solution of the $\overline{\partial}$-Neumann problem (see [K1], [K2] and also [KN]), which implies existence and regularity results for the $\overline{\partial}$-complex, as well (see Chapter III.1 in [FK]). But whereas the theory is very well developed on complex manifolds, it has been an open problem ever since to create an appropriate L^2-theory for the $\overline{\partial}$-operator on singular complex spaces. We will give a partial answer to that problem in the present paper.

When we consider the $\overline{\partial}$-operator on singular complex spaces, the first problem is to define an appropriate Dolbeault complex in the presence of singularities. It turns out that it is very fruitful to investigate the $\overline{\partial}$-operator in the L^2-category (simply) on the complex manifold consisting of the regular points of a complex space. One reason lies in Goresky and MacPherson's notion of intersection (co-)homology (see [GM1, GM2]) and the conjecture of Cheeger, Goresky and MacPherson, which states that the L^2-deRham cohomology on the regular part of a projective variety Y (with respect to the restriction of the Fubini-Study

[1]For the further development, we also recommend [D1].

metric and the exterior derivate in the sense of distributions) is naturally isomorphic to the intersection cohomology of middle perversity $IH^*(Y)$ of Y:

Conjecture 1.1. (Cheeger-Goresky-MacPherson [CGM])
Let $Y \subset \mathbb{CP}^N$ be a projective variety. Then there is a natural isomorphism

$$H_{(2)}^k(Y - \mathrm{Sing}\,Y) \cong IH^k(Y). \tag{1}$$

The early interest in this conjecture was motivated in large parts by the hope that one could then use the natural isomorphism and a classical Hodge decomposition for $H_{(2)}^k(Y - \mathrm{Sing}\,Y)$ to put a pure Hodge structure on the intersection cohomology of Y (see [PS2]). We will discuss that point at the end of the introduction. Note that in this setting, the singular set appears as a (quite irregular) boundary of the domain where we consider the $\overline{\partial}$-equation.

It is also interesting to have a look at the arithmetic genus of complex varieties. When M is a compact complex manifold, the arithmetic genus

$$\chi(M) := \sum (-1)^q \dim H^{0,q}(M) \tag{2}$$

is a birational invariant of M. The conjectured extension of the classical Hodge decomposition to projective varieties led MacPherson also to ask wether the arithmetic genus $\chi(M)$ extends to a birational invariant of all projective varieties (see [M]). To formulate MacPherson's question slightly more general, we call a reduced and paracompact complex space X Hermitian if the regular part $X - \mathrm{Sing}\,X$ carries a Hermitian metric which is locally the restriction of a Hermitian metric in some complex number space where X is represented locally. For example, a projective variety is Hermitian with the the restriction of the Fubini-Study metric.

Conjecture 1.2. (MacPherson) *If X is a Hermitian compact complex space, then*

$$\chi_{(2)}(X - \mathrm{Sing}\,X) := \sum (-1)^q \dim H_{(2)}^{0,q}(X - \mathrm{Sing}\,X) = \chi(M),$$

where $\pi : M \to X$ is any resolution of singularities.

Due to the incompleteness of the metric on $X - \mathrm{Sing}\,X$, one has to be very careful with the definition of Dolbeault cohomology groups $H_{(2)}^{0,q}$ for they depend on the choice of some kind of boundary condition for the $\overline{\partial}$-operator. To explain that more precisely, let $\overline{\partial}_c$ be the $\overline{\partial}$-operator acting on smooth forms with compact support away from $\mathrm{Sing}\,X$:

$$\overline{\partial}_c : \ A_{cpt}^{p,q}(X - \mathrm{Sing}\,X) \to A_{cpt}^{p,q+1}(X - \mathrm{Sing}\,X).$$

We may consider $\overline{\partial}_c$ as an operator acting on square-integrable forms:

$$\overline{\partial}_c : \ \mathrm{Dom}\,\overline{\partial}_c = A_{cpt}^{p,q}(X - \mathrm{Sing}\,X) \subset L_{p,q}^2(X - \mathrm{Sing}\,X) \to L_{p,q+1}^2(X - \mathrm{Sing}\,X).$$

This operator has various closed extensions. The two most important extensions are the minimal closed extension, namely the closure of the graph of $\overline{\partial}_c$ in

$$L^2_{p,q}(X - \operatorname{Sing} X) \times L^2_{p,q+1}(X - \operatorname{Sing} X)$$

which we will denote by $\overline{\partial}_s$, and the maximal closed extension, that is the $\overline{\partial}$-operator in the sense of distributions which we will denote by $\overline{\partial}_w$.[2] It follows from the results below that they lead to different Dolbeault cohomology groups which we will call $H^{p,q}_s(X - \operatorname{Sing} X)$ and $H^{p,q}_w(X - \operatorname{Sing} X)$, respectively.[3]

MacPherson's Conjecture 1.2 has been settled for projective varieties by Pardon and Stern [PS1] for the arithmetic genus with respect to $\overline{\partial}_s$:

$$\chi_s(X - \operatorname{Sing} X) := \sum (-1)^q H^{0,q}_s(X - \operatorname{Sing} X).$$

Rather than comparing alternating sums, they realized that the groups $H^{0,q}_s$ themselves are birational invariants:

Theorem 1.3. (Pardon-Stern [PS1]**)** *If Y is a complex projective variety of pure dimension n and $Y - \operatorname{Sing} Y$ is given the Hermitian metric induced by the embedding of Y in projective space, then the groups $H^{0,q}_s(Y - \operatorname{Sing} Y)$ are birational invariants of Y, and in fact, for $0 \le q \le n$,*

$$H^{0,q}_s(Y - \operatorname{Sing} Y) \cong H^{0,q}(M),$$

where $\pi : M \to Y$ is any resolution of singularities.

For the $\overline{\partial}$-operator in the sense of distributions, they claim the following:

Theorem 1.4. (Pardon-Stern [PS1]**)** *For Y as in Theorem 1.3 with isolated singularities only, $\dim Y \le 2$, and $0 \le q \le 2$,*

$$H^{0,q}_w(Y - \operatorname{Sing} Y) \cong H^q(M, \mathcal{O}(Z - |Z|)), \tag{3}$$

where $\pi : M \to Y$ is a resolution of singularities with only normal crossings, and Z the unreduced exceptional divisor $Z = \pi^{-1}(\operatorname{Sing} Y)$.[4]

The work of Pardon and Stern [PS1] marks the interim point of culmination of the research about the $\overline{\partial}$-operator on projective complex spaces. Earlier contributions towards Theorem 1.3 and Theorem 1.4 had been made by Haskell [H1], Pardon [P] and Nagase [N2] in the special cases of complex curves and surfaces.

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[2] Our subscript refers to $\overline{\partial}_s$ as an extension in a strong sense and to $\overline{\partial}_w$ as an extension in a weak sense. Note that Pardon and Stern use the notation $\overline{\partial}_D$ for $\overline{\partial}_s$ and $\overline{\partial}_N$ for $\overline{\partial}_w$. Their notation refers to some kind of Dirichlet respectively Neumann boundary conditions (see [PS1]). It is interesting to study under which circumstances the extensions coincide. See the work of Grieser and Lesch [GL], Pardon and Stern [PS2], or Brüning and Lesch [BL] for this topic.

[3] This phenomenon occurs also in other singular configurations (see [BS]).

[4] It seems that Theorem 1.4 is only partially proven until now because there are some doubts about Lemma (3.6), precisely the estimate (3.7), in [PS1]. So, we might consider Theorem 1.4 as a conjecture.

In the present paper, we continue the work of Pardon and Stern by generalizing both, Theorem 1.3 and Theorem 1.4, to arbitrary Hermitian compact complex spaces. Concerning Theorem 1.3, we only have to replace the application of the Grauert-Riemenschneider vanishing theorem [GR] by Takegoshi's relative version [T] in the proof of Pardon and Stern and obtain:

Theorem 1.5. *Let X be a Hermitian compact complex space of pure dimension, and $0 \leq q \leq n = \dim X$. Then*

$$H_s^{0,q}(X - \operatorname{Sing} X) \cong H^{0,q}(M) \tag{4}$$

for any resolution of singularities $\pi : M \to X$.

Note that this settles MacPherson's Conjecture 1.2 for Hermitian compact complex spaces. Concerning Theorem 1.4, the situation is much more sophisticated. If Y is a complex surface with isolated singularities, one is in a very special situation where one can use Hsiang-Pati coordinates for a resolution of singularities (see [HP]).[5] In general, one has to develop completely new techniques. The general philosophy is to compare the boundary behavior of $\overline{\partial}_w$- and $\overline{\partial}_s$-harmonic[6] forms, and to link statements about $H_w^{0,q}(X - \operatorname{Sing} X)$ to a version of Theorem 1.5 for forms with values in a certain holomorphic line bundle (Theorem 4.1). As in Theorem 1.4, let $\pi : M \to X$ be a resolution of singularities with only normal crossings, and Z the unreduced exceptional divisor $Z := \pi^{-1}(\operatorname{Sing} X)$. Now, let $L \to M$ be the Hermitian holomorphic line bundle associated to $Z - |Z|$ such that $\mathcal{O}(L) \cong \mathcal{O}(Z - |Z|)$. We denote by $\pi_* L$ the push-forward to $X - \operatorname{Sing} X$ of the restriction of L to $M - |Z|$. We will show that

$$H_w^{0,q}(X - \operatorname{Sing} X) \cong H_s^{0,q}(X - \operatorname{Sing} X, \pi_* L) \tag{5}$$

if X has only isolated singularities under the assumption that L is semi-negative (Theorem 6.4). That explains the general idea behind our main result:

Theorem 1.6. *Let X be a Hermitian compact complex space of pure dimension with isolated singularities, $\pi : M \to X$ a resolution of singularities with only normal crossings, and $0 \leq q \leq n = \dim X$. If the line bundle associated to $Z - |Z|$ is locally semi-negative with respect to the base-space X, then*

$$H_w^{0,q}(X - \operatorname{Sing} X) \cong H^q(M, \mathcal{O}(Z - |Z|)), \tag{6}$$

where Z is the unreduced exceptional divisor $Z = \pi^{-1}(\operatorname{Sing} X)$.[7]

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[5] The case $\dim Y = 1$ is even simpler.

[6] We call a differential form $\overline{\partial}_w$-harmonic (resp. $\overline{\partial}_s$-harmonic) if it is $\overline{\partial}_w$- and $\overline{\partial}_w^*$-closed (resp. $\overline{\partial}_s$- and $\overline{\partial}_s^*$-closed).

[7] It is required that each point $p \in X$ has a neighborhood U_p such that $Z - |Z|$ is semi-negative on $\pi^{-1}(U_p)$. This assumption is fulfilled e.g. if $\dim X = 1$ or if the divisor Z has the same order on each of its irreducible components. That happens for example at conical singularities where the resolution is obtained by a single blow-up.

It is worthwhile to mention that compactness of M implies that the cohomology groups in (4) and (6) are finite-dimensional.

By Serre duality and its L^2-version (Theorem 2.3), (4) is equivalent to

$$H_w^{n,q}(X - \operatorname{Sing} X) \cong H^{n,q}(M), \ 0 \leq q \leq n, \tag{7}$$

whereas (6) is equivalent to

$$H_s^{n,q}(X - \operatorname{Sing} X) \cong H^q(M, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)), \ 0 \leq q \leq n. \tag{8}$$

Here, $\mathcal{K}_M$ denotes the canonical sheaf on M.

Let us recall the main points of the proof of Theorem 1.3 from the work of Pardon and Stern, but in the situation of Theorem 1.5. Let $\pi : M \to X$ be a resolution of singularities. The key point is to consider the canonical sheaf $\mathcal{K}_M$ because $(n, 0)$-forms behave very well under resolution of singularities. Let $\mathcal{A}_M^{n,*}$ be the $\overline{\partial}$-complex of sheaves of germs of smooth $(n, *)$-forms which is a fine resolution for $\mathcal{K}_M$. Takegoshi's generalization [T] of the Grauert-Riemenschneider vanishing theorem [GR] states that the higher direct image sheaves of the canonical sheaf on M do vanish:

$$R^q \pi_* \mathcal{K}_M = 0 \ , \ q > 0.$$

This implies that the direct image complex $\pi_* \mathcal{A}_M^{n,*}$ is still exact, and so it gives a fine resolution of $\pi_* \mathcal{K}_M$, and we obtain:

$$H^q(X, \pi_* \mathcal{K}_M) \cong H^q(M, \mathcal{K}_M) \cong H^{n,q}(M) \cong H^{0,n-q}(M). \tag{9}$$

On the other hand, the natural sheafification of the L^2-$\overline{\partial}$-complex on X is also a fine resolution of $\pi_* \mathcal{K}_M$. Exactness for that complex follows from a local vanishing result first proved by Ohsawa in case X has only isolated singularities [O1]. The idea, which appears with essentially the same argument in an earlier paper of Demailly [D1], is to approximate the incomplete metric by a sequence of complete metrics for which one already knows the local vanishing result by a theorem of Donelly and Fefferman [DF]. The cohomology of global sections of this complex is $H_w^{n,*}(X - \operatorname{Sing} X)$, and the exactness yields:

$$H^q(X, \pi_* \mathcal{K}_M) \cong H_w^{n,q}(X - \operatorname{Sing} X).$$

Now, the L^2-version of Serre duality

$$H_w^{n,q}(X - \operatorname{Sing} X) \cong H_s^{0,n-q}(X - \operatorname{Sing} X)$$

and (9) yield Theorem 1.5.

For a sketch of the proof of Theorem 1.6, let $\pi : M \to X$ be a resolution of singularities with only normal crossings, and let Z be the unreduced exceptional divisor $Z = \pi^{-1}(\operatorname{Sing} X)$. Similarly to Theorem 1.5, we will prove Theorem 1.6 by giving two fine resolutions for the direct image sheaf $\pi_* \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)$. The first resolution is given by the direct image complex $\pi_* \mathcal{A}_M^{n,*} \otimes \mathcal{O}(|Z| - Z)$ which

is exact by Takegoshi's vanishing theorem if $\mathcal{O}(|Z| - Z)$ is locally semi-positive with respect to the base-space X (see section 4). Hence, as in (9):

$$H^q(X, \pi_* \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \cong H^q(M, \mathcal{O}(Z - |Z|)). \qquad (10)$$

On the other hand, the natural sheafification of the L^2-$\overline{\partial}_s$-complex on X is also a fine resolution of $\pi_* \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)$ (see section 6), yielding

$$H^q(X, \pi_* \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \cong H^{n,q}_s(X - \operatorname{Sing} X) \cong H^{0,q}_w(X - \operatorname{Sing} X).$$

To prove local exactness of the $\overline{\partial}_s$-complex, it is essential to notice that $Z - |Z|$ describes the behavior of the reduction $\pi : M \to X$ in directions normal to the exceptional set, and to use harmonic representation and the Cauchy formula in one complex variable. Local $\overline{\partial}_s$-exactness for forms of degree $q \geq 1$ follows from regularity results for the L^2-$\overline{\partial}$-equation at isolated singularities of Fornæss, Øvrelid and Vassiliadou (Theorems 1.1 and 1.2 in [FOV2]).

Note that $H^{0,q}_s(X - \operatorname{Sing} X) \cong H^{0,q}_w(X - \operatorname{Sing} X)$ on a space with isolated singularities if $H^q(M, \mathcal{O}) \cong H^q(M, \mathcal{O}(Z - |Z|))$, which might well occur if $Z = |Z|$ or $q < n - 1$. See also [PS2] and [GL], [BL] for this topic.

It seems appropriate to make some remarks about the relation of the present paper to intersection (co-)homology. Ohsawa proved the Conjecture of Cheeger, Goresky and MacPherson under the assumption that Y has only isolated singularities (see [O2]), while it is still open in general. Ohsawa's proof depends on an earlier work of Saper who constructed a complete metric on $Y - \operatorname{Sing} Y$ such that the L^2-deRham cohomology in this complete metric equals the intersection cohomology of Y (see [S2]). Ohsawa used a family of such complete metrics which degenerate to the incomplete restriction of the Fubini-Study metric on $Y - \operatorname{Sing} Y$, and showed that the L^2-cohomology is stable under the limit process by the use of strong L^2-estimates going back to Donnelly and Fefferman (see [DF]).

The Conjecture of Cheeger, Goresky and MacPherson is closely related to the (more famous) Zucker Conjecture which states that the L^2-cohomology of a Hermitian locally symmetric space is isomorphic to the intersection cohomology of middle perversity of its Baily-Borel compactification [Z1]. Zucker's conjecture was proved independently by Looijenga [L] and by Saper and Stern [SS].

The early interest in Conjecture 1.1 was motivated in large parts by the hope that one could then use the isomorphism (1) and a Hodge decomposition for

$$H^k_{(2)}(Y - \operatorname{Sing} Y)$$

to put a pure Hodge structure on the intersection cohomology of Y (see [CGM]).

Eventually, using methods not directly related to L^2-cohomology (D-modules), Saito established a pure Hodge decomposition in the sense of Deligne for the intersection cohomology of singular varieties ([S1]). However, it still seems interesting to get a classical Hodge decomposition on $H^k_{(2)}(Y - \operatorname{Sing} Y)$ and to investigate the relation to Saito's construction via an isomorphism (1). This was done in special

cases by Zucker [Z2] and Ohsawa [O3]. In the case of isolated singularities, a decomposition of $H_{(2)}^k(Y - \operatorname{Sing} Y)$ in terms of Dolbeault cohomology groups was in fact established by Pardon and Stern (in [PS2]). Combined with Ohsawa's solution of the Cheeger-Goresky-MacPherson Conjecture, this gives a Hodge decomposition for the intersection cohomology on such varieties. See also the work of Nagase [N1] and Fox and Haskell [FH] for related decomposition results on projective normal complex surfaces.

We should also give a short account about the state of the art of the research about the $\overline{\partial}$-operator on singular complex spaces in general. First steps towards an L^2-theory for the $\overline{\partial}$-operator on singular complex spaces had been made by Fornæss [F], Diederich, Fornæss and Vassiliadou [DFV], Fornæss, Øvrelid and Vassiliadou [FOV1, FOV2], and Øvrelid and Vassiliadou [OV1, OV2, OV3]. All these works have in common that they do basically use L^2-methods to prove some finiteness results for L^2-Dolbeault cohomology groups.

On the contrary, in the present paper, we focus on computing the cohomology explicitly in terms of a resolution of singularities.[8] This agenda has been already pursued in Ruppenthal [R2, R3], where we do not only use L^2-methods, but also integral formulas on analytic varieties. Integral formulas have the advantage that they also yield L^p-results and statements about Hölder regularity of the $\overline{\partial}$-equation. Slightly more general integral formulas on weighted homogeneous varieties have been established by Ruppenthal and Zeron [RZ1, RZ2], proving L^2- and L^p-results, but also optimal Hölder estimates on such varieties.

Concerning the research on Hölder regularity of the $\overline{\partial}$-equation on singular spaces, me must also mention the initial work of Fornæss and Gavosto [FoGa] which was generalized and extended by Ruppenthal [R1], and the treatment of rational double points[9] by Acosta and Zeron [AZ1, AZ2] and Solis and Zeron [SZ]. Whereas Fornæss, Gavosto and Ruppenthal use a quite general approach based on finitely-sheeted analytic coverings, the work of Acosta, Solis and Zeron relies on parametrization by Stein spaces which is restricted to special situations.

For the sake of completeness, we also mention the work of Berndtsson and Sibony [BS] about the $\overline{\partial}$-equation on positive currents[10], and the integral formulas on analytic varieties of Henkin and Polyakov [HP], generalized by Andersson and Samuelsson recently [AS].

The present paper is organized as follows: in section 2, we introduce the basic technical concepts and review some background results from functional analysis, including harmonic representation of cohomology classes which is at the core of our methods. In section 3, we prove Theorem 1.5 by reviewing the proof of

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[8]Acknowledging that we use regularity results of Fornæss, Øvrelid and Vassiliadou ([FOV2], Theorem 1.1 and Theorem 1.2) in the proof of Theorem 1.6.

[9]Isolated singularities of hypersurfaces in $\mathbb{C}^3$.

[10]Berndtsson and Sibony also observe a different behavior of the $\overline{\partial}_w$- and the $\overline{\partial}_s$-operator which is quite analogous to our results.

Theorem 1.3 of Pardon and Stern (and replacing the Grauert-Riemenschneider vanishing theorem by Takegoshi's relative version), while we prove a version of Theorem 1.5 for forms with values in certain holomorphic line bundles (Theorem 4.1) in section 4. As a preparation for the proof of Theorem 1.6, we study different kinds of L^2-Dolbeault cohomology groups at isolated singularities in section 5. In section 6, we define the $\overline{\partial}_s$-complex on a resolution $\pi : M \to X$ and show that its direct image complex is exact on X, yielding Theorem 1.6. This also allows to relate $\overline{\partial}_w$- and $\overline{\partial}_s$-Dolbeault cohomology (Theorem 6.4) by combining Theorem 1.6 with Theorem 4.1. In the last section, we observe that our methods are of local nature and can be used to study the $\overline{\partial}$-equation also in a local sense if we avoid boundary problems (Theorem 7.1).

As an introduction, we do also strongly recommend the paper of Pardon and Stern [PS1] which is the starting point of our present work.

2. Harmonic Representation

For the basic definitions and a few background results from functional analysis, we mainly follow the exposition in [PS1] and adopt most of their notations. We also refer to the harmonic theory on compact complex manifolds as it is presented in [W], and to Demailly's introduction to Hodge theory [D2]. It is crucial to observe that the Dolbeault cohomology classes we are interested in have unique harmonic representatives, yielding important duality relations.

If M is a complex manifold, γ will always denote a positive semidefinite Hermitian metric on M which is generically (i.e. almost everywhere) definite. So, there is a closed subset E of M such that γ is positive definite on the complement of E, and the smallest such subset is called the degeneracy locus of γ.

We consider the following situation. Let (X, h) be a Hermitian compact complex space of pure dimension n, and

$$\pi : M \to X$$

a resolution of singularities (which exists by Hironaka [H3]), i.e. a proper holomorphic surjection such that

$$\pi|_{M-E} : M - E \to X - \operatorname{Sing} X$$

is biholomorphic, where $E = |\pi^{-1}(\operatorname{Sing} X)|$ is the exceptional set as a reduced complex space. We assume that E is a divisor with only normal crossings, but remark that this assumption is not necessary for the proof of Theorem 1.5. For the topic of desingularization, we refer to [AHL], [BM] and [H2]. Now then let

$$\gamma := \pi^* h$$

be the pullback of the Hermitian metric h to M. γ is positive semidefinite with degeneracy locus E.

A positive semidefinite Hermitian metric γ induces a pointwise inner product on (p, q)-forms α and β (which is positive definite almost everywhere and has certain poles on the degeneracy locus)

$$\langle \alpha, \beta \rangle_\gamma, \ |\alpha|_\gamma := \sqrt{\langle \alpha, \alpha \rangle},$$

and gives an inner product on the vector space of (p, q)-forms

$$
\begin{aligned}
(\alpha, \beta)_\gamma &:= \int_M \langle \alpha, \beta \rangle_\gamma dV_\gamma = \int_M \alpha \wedge *_\gamma \beta, \\
\|\alpha\|_\gamma &:= \sqrt{(\alpha, \alpha)_\gamma},
\end{aligned}
$$

where dV_γ is the volume form of γ, and $*_\gamma$ is the Hodge-$*$-operator with respect to γ which is defined almost everywhere for the time being. The resulting norm $\| \cdot \|_\gamma$ is the L^2_γ-norm in the sequel. The description of $\langle \cdot, \cdot \rangle_\gamma$ and of the action of $*_\gamma$ are rather complicated for forms of arbitrary degree. However, it is well understood for functions and for forms of type $(n, 0)$, $(0, n)$ and (n, n).

Let us compare dV_γ to the volume form of any positive definite Hermitian metric σ on M. Let p be a point in M and U_p a neighborhood of p such that we can assume that $\pi(U_p)$ is embedded holomorphically in some complex number space $\mathbb{C}^L$, $L \gg n$,

$$\pi|_{U_p} : U_p \to \pi(U_p) \subset \mathbb{C}^L$$

with holomorphic coordinates $z_1, ..., z_n$ in U_p, and that h is the restriction of the Euclidean metric in $\mathbb{C}^L$ to $\operatorname{Reg} X$. Let dV_h be the volume form with respect to the metric h on $\pi(U_p) \cap (X - \operatorname{Sing} X)$. After shrinking U_p if necessary, we can assume that there exists a holomorphic function $J \in \mathcal{O}(U_p)$ such that

$$|J|^2 = \det \operatorname{Jac}_\mathbb{R} \pi = \det \left({}^t\overline{\operatorname{Jac}_\mathbb{C} \pi} \cdot \operatorname{Jac}_\mathbb{C} \pi \right) \tag{11}$$

on U_p and that the zero set of J is contained in the exceptional set E because π is biholomorphic outside E. For a proof, we refer to [R2], Lemma 2.1. (11) has to be understood in the following sense: $\pi : U_p \setminus E \to \operatorname{Reg} X$ is a diffeomorphism which has a well defined determinant of the real Jacobian $\operatorname{Jac}_\mathbb{R} \pi$ that extends as $|J|^2$ to U_p. By choosing local holomorphic coordinates on $\operatorname{Reg} X$ and using (15) from below, we get the right hand side of (11) where $\operatorname{Jac}_\mathbb{C} \pi$ is the complex Jacobian of π as a mapping $\pi : \mathbb{C}^n \to \mathbb{C}^L$. It follows that

$$dV_\gamma = \pi^* dV_h = (\det \operatorname{Jac}_\mathbb{R} \pi)\, dV_\sigma = |J|^2 dV_\sigma \tag{12}$$

So there exists a non-vanishing holomorphic $(n, 0)$-form ω on

$$\pi(U_p) \cap \big(X - \operatorname{Sing} X \big)$$

such that

$$\pi^* \omega = J \cdot dz_1 \wedge \cdots \wedge dz_n \tag{13}$$

with

$$|\omega|_h = |\pi^* \omega|_\gamma \equiv 1.$$

It follows that

$$\omega \wedge \omega = dV_h, \quad \pi^* \omega \wedge \pi^* \omega = dV_\gamma = \pi^* dV_h. \tag{14}$$

If f is a function on U_p, we obtain with (12), (13) and (14):[11]

$$
\begin{aligned}
*_\gamma f &= f dV_\gamma = f|J|^2 dV_\sigma, \\
*_\gamma (f dV_\sigma) &= |J|^{-2} f *_\gamma (|J|^2 dV_\sigma) = |J|^{-2} f, \\
*_\gamma \big(f dz_1 \wedge \cdots \wedge dz_n \big) &= *_\gamma \big((fJ^{-1}) J dz_1 \wedge \cdots \wedge dz_n \big) \\
&= (fJ^{-1}) *_\gamma \big(J dz_1 \wedge \cdots \wedge dz_n \big) \\
&\sim (fJ^{-1}) J dz_1 \wedge \cdots \wedge dz_n \\
&= f dz_1 \wedge \cdots \wedge dz_n.
\end{aligned}
$$

Since M is compact, that yields:

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[11] We write $\omega \sim \eta$ for two forms ω, η if there exists a continuous function f such that $\omega = f\eta$, and f is bounded and bounded away from zero.

Lemma 2.1. *If σ is any positive definite Hermitian metric on M, and η a $(n,0)$ or $(0,n)$-form, then*

$$\langle \eta, \eta \rangle_\gamma dV_\gamma = \eta \wedge *_\gamma \eta \quad \sim \quad \eta \wedge *_\sigma \eta = \langle \eta, \eta \rangle_\sigma dV_\sigma.$$

Thus $L_\gamma^{n,0} \cong L_\sigma^{n,0}$ and $L_\gamma^{0,n} \cong L_\sigma^{0,n}$ for the L^2-norms on $(n,0)$ and $(0,n)$-forms.

For later use we have to study some further properties of the metric $\gamma = \pi^* h$ on an open set U_p chosen as above,

$$\pi = (\pi_1, ..., \pi_L) : U_p \subset \mathbb{C}^n \to X \subset \mathbb{C}^L,$$

and we can still assume that h is the restriction of the Euclidean metric to $\operatorname{Reg} X$. Then $\gamma = \pi^* h$ is represented by the matrix $G = (g_{jk})_{j,k=1}^n$ with coefficients

$$g_{jk} = \langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \rangle_\gamma = \langle \pi_* \frac{\partial}{\partial z_j}, \pi_* \frac{\partial}{\partial z_k} \rangle_h = \langle \pi_* \frac{\partial}{\partial z_j}, \pi_* \frac{\partial}{\partial z_k} \rangle_{\mathbb{C}^L} \tag{15}$$

because the vectors $\pi_* \frac{\partial}{\partial z_j}$ are tangential to $\operatorname{Reg} X$. Thus

$$G = {}^t\overline{\operatorname{Jac}_\mathbb{C} \pi} \cdot \operatorname{Jac}_\mathbb{C} \pi, \quad \det G = |J|^2.$$

Note that the coefficients of G are real-analytic. In order to compute $|dz_L|_\gamma$ for a multi-index $L = (l_1, ..., l_q)$, let $G^{-1} = (g^{jk})$ be the inverse matrix and

$$G^{-1} = \frac{1}{\det G} H = \frac{1}{|J|^2} H, \tag{16}$$

where $H = (h^{jk}) = \det G \cdot G^{-1}$ is a matrix with real-analytic coefficients (the adjugate matrix). Then:

$$\langle dz_j, dz_k \rangle_\gamma = g^{jk} = |J|^{-2} h^{jk}. \tag{17}$$

For the multi-index $L = (l_1, ..., l_q)$, $l_1 < ... < l_q$, let H^L be the $(q \times q)$-submatrix of H that we get by taking the lines $l_1, ..., l_q$ and the rows $l_1, ..., l_q$. It follows from (16) and (17) by standard linear algebra that

$$|d\overline{z_L}|_\gamma^2 = |dz_L|_\gamma^2 = \frac{1}{|J|^{2q}} \det H^L, \tag{18}$$

where $\det H^L$ is a real-analytic function since the coefficients of H^L are real-analytic.

We need to consider more generally differential forms with values in holomorphic line bundles associated to divisors with support contained in the exceptional set E of the desingularization. Fix a locally finite defining system $\{(U_\alpha, f_\alpha)\}$ for the divisor E such that there exists a smooth partition of unity $\{\phi_\alpha\}$ subordinate to $\{U_\alpha\}$. Keep this system fixed throughout the paper. Since E has normal crossings only, we can assume that f_α is of the form

$$f_\alpha = z_1 \cdots z_m,$$

with $m \leq n$ depending on α, and for a multi-index $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, let

$$f_\alpha^k = z_1^{k_1} \cdots z_m^{k_m}.$$

Let Z be a divisor on M with support contained in E, i.e. Z is defined by a system $\{(U_\alpha, f_\alpha^{k_\alpha(Z)})\}$ with coefficients $k_\alpha(Z) \in \mathbb{Z}^n$. Let L_Z be the unique holomorphic line bundle associated to Z with transition functions

$$g_{\alpha\beta} = f_\alpha^{k_\alpha(Z)} f_\beta^{-k_\beta(Z)} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

We will also use the dual bundle $L_Z^* = L_{-Z}$ given by the inverse transition functions $g_{\alpha\beta}^{-1}$. For an open set $U \subset M$, the space of smooth (p,q)-forms on U with values in L_Z is

$$A^{p,q}(U, L_Z) = \Gamma(U, \Lambda^{p,q} T^* M \otimes L_Z).$$

This defines sheaves $\mathcal{A}^{p,q}(L_Z)$ which are canonically isomorphic to $\mathcal{A}^{p,q} \otimes \mathcal{O}(Z)$. Due to the special structure of L_Z, we have

$$A^{p,q}(U, L_Z) \cong \{\eta \text{ measurable } (p,q)\text{-form} | \forall \alpha : f_\alpha^{k_\alpha(Z)} \eta \in A^{p,q}(U \cap U_\alpha)\}. \quad (19)$$

When we deal with forms with values in L_Z, we will usually use this representation. We give L_Z the structure of a Hermitian holomorphic line bundle by defining the pointwise inner product on (p,q) forms with values in L_Z as

$$\langle \eta, \psi \rangle_{Z,\gamma} \;:=\; \sum_\alpha \phi_\alpha \langle f_\alpha^{k_\alpha(Z)} \eta, f_\alpha^{k_\alpha(Z)} \psi \rangle_\gamma = \sum_\alpha \phi_\alpha |f_\alpha^{k_\alpha(Z)}|^2 \langle \eta, \psi \rangle_\gamma, \quad (20)$$

$$|\eta|_{Z,\gamma} \;:=\; \sqrt{\langle \eta, \eta \rangle_\gamma}. \quad (21)$$

Clearly that construction depends on the choice of the system $\{(U_\alpha, f_\alpha, \phi_\alpha)\}$, but M is compact, and so different defining systems for E give equivalent metrics. Note that the inner product dual to $\langle \cdot, \cdot \rangle_{Z,\gamma}$ on the dual bundle L_Z^* is just $\langle \cdot, \cdot \rangle_{-Z,\gamma}$.

We will use the definition of $\langle \cdot, \cdot \rangle_{Z,\gamma}$ and all the following constructions either for $\gamma = \pi^* h$ or any positive definite Hermitian metric σ on M. As a connection on L_Z we use the Chern connection $D = D' + D'' = D' + \overline{\partial}$ with respect to the metric $\langle \cdot, \cdot \rangle_{Z,\gamma}$.

Our next purpose is to define the Hodge-$*$-operator for differential forms with values in L_Z. It is more convenient to work with the conjugate-linear operator

$$\overline{*}_\gamma \eta := *_\gamma \overline{\eta}.$$

Let $\tau : L_Z \to L_Z^* = L_{-Z}$ be the canonical conjugate-linear bundle isomorphism of L_Z onto its dual bundle. τ acts on a smooth section

$$g \in \Gamma(U, L_Z) \cong \{u \text{ measurable function} | \forall \alpha : f_\alpha^{k_\alpha(Z)} u \in A^{0,0}(U \cap U_\alpha)\}$$

as

$$\tau(g) = \sum_\alpha \phi_\alpha |f_\alpha^{k_\alpha(Z)}|^2 g \quad \in \Gamma(U, L_Z^*),$$

because $\tau(g)$ is the section in L_Z^* such that

$$\tau(g)(h) = \tau(g) \cdot h = \langle h, g \rangle_{Z,\gamma} = \sum_\alpha \phi_\alpha |f_\alpha^{k_\alpha(Z)}|^2 gh$$

for all $h \in \Gamma(U, L_Z)$. We can now define the conjugate-linear isomorphism

$$*_{Z,\gamma} : \Lambda^{p,q}T^*M \otimes L_Z \to \Lambda^{n-p,n-q}T^*M \otimes L_Z^*$$

almost everywhere by setting

$$*_{Z,\gamma}(\eta \otimes e) := *_\gamma \eta \otimes \tau(e).$$

This gives the following representation for the inner product on (p, q)-forms with values in L_Z:

$$(\eta, \psi)_{Z,\gamma} := \int_M \langle \eta, \psi \rangle_{Z,\gamma} dV_\gamma = \int_M \eta \wedge *_{Z,\gamma} \psi,$$

$$\|\eta\|_{Z,\gamma} := \sqrt{(\eta, \eta)_{Z,\gamma}}.$$

The resulting norm $\| \cdot \|_{Z,\gamma}$ is the $L_\gamma^2(Z)$-norm in the sequel. We remark that the operator $*_{Z,\gamma}$ is denoted $\#$ in Demailly's introduction to Hodge theory [D2].

The action of the $\overline{\partial}$-operator is well-defined for forms with values in L_Z. If we use the representation (19) for such a form $\eta \in A^{p,q}(U, L_Z)$, then $D''\eta = \overline{\partial}\eta \in A^{p,q+1}(U, L_Z)$ is given locally as

$$\overline{\partial}\eta = f_\alpha^{-k_\alpha(Z)} \overline{\partial} \left(f_\alpha^{k_\alpha(Z)} \eta \right).$$

Suppose now that $\eta \in A^{p,q-1}(M, L_Z)$ and $\psi \in A^{p,q}(M, L_Z)$ with compact support in $M - E$. So, $\eta \wedge *_{Z,\gamma} \psi$ is a scalar valued $(n, n-1)$-form and it is easy to compute:

$$
\begin{aligned}
d(\eta \wedge *_{Z,\gamma} \psi) &= \overline{\partial}(\eta \wedge *_{Z,\gamma} \psi) \\
&= \overline{\partial}\eta \wedge *_{Z,\gamma} \psi + (-1)^{p+q-1} \eta \wedge \overline{\partial}(*_{Z,\gamma} \psi).
\end{aligned}
$$

It follows by Stokes Theorem that

$$
\begin{aligned}
(\overline{\partial}\eta, \psi)_{Z,\gamma} &= (-1)^{p+q} \int_M \eta \wedge \overline{\partial} *_{Z,\gamma} \psi \\
&= -\int_M \eta \wedge *_{Z,\gamma} *_{-Z,\gamma} \overline{\partial} *_{Z,\gamma} \psi \\
&= (\eta, -*_{-Z,\gamma} \overline{\partial} *_{Z,\gamma} \psi)_{Z,\gamma}.
\end{aligned}
$$

Thus, we note:

Lemma 2.2. *The formal adjoint of the $\overline{\partial}$-operator for forms with values in the holomorphic line bundle L_Z with respect to the $L^2_\gamma(Z)$-norm is*

$$\vartheta := -*_{-Z,\gamma}\overline{\partial}*_{Z,\gamma}. \tag{22}$$

We define now the relevant sheaves analogously to [PS1]. Again, we may as well allow any positive definite metric σ on M instead of $\gamma = \pi^*h$ in the definitions.

(1) $\mathcal{L}^{p,q}_\gamma(L_Z) :=$ the sheaf of germs of measurable (p,q)-forms with values in L_Z that are locally $L^2_\gamma(Z)$.
(2) $\mathcal{S}^{p,q}_{\gamma,E}(L_Z) := \mathcal{L}^{p,q}_\gamma(L_Z) \cap \mathcal{A}^{p,q}_{M-E}(L_Z)$, where $\mathcal{A}^{p,q}_{M-E}$ is the sheaf of germs of smooth (p,q)-forms with values in L_Z on $M - E$.

The corresponding $\overline{\partial}$-complexes of sheaves on M are, for each p:

(1) $(\mathcal{C}^{p,*}_{\gamma,E}(L_Z), \overline{\partial}_w)$, where

$$\mathcal{C}^{p,q}_{\gamma,E}(L_Z) := \mathcal{L}^{p,q}_\gamma(L_Z) \cap \overline{\partial}_w^{-1}\mathcal{L}^{p,q+1}_\gamma(L_Z),$$

and $\overline{\partial}_w$ is the (weak) $\overline{\partial}$-operator in the sense of distributions on L_Z with respect to compact subsets of $M - E$.
(2) $(\mathcal{A}^{p,*}_{\gamma,E}(L_Z), \overline{\partial})$, where

$$\mathcal{A}^{p,q}_{\gamma,E}(L_Z) := \mathcal{S}^{p,q}_{\gamma,E}(L_Z) \cap \overline{\partial}^{-1}\mathcal{S}^{p,q+1}_{\gamma,E}(L_Z).$$

The global sections of a sheaf will be denoted by the corresponding upper latin case character, e.g.

$$A^{p,q}_{\gamma,E}(M, L_Z) := \Gamma(M, \mathcal{A}^{p,q}_{\gamma,E}(L_Z)).$$

Moreover, let

$$A^{p,q}_{cpt}(M - E)$$

be the smooth compactly supported (p,q)-forms on $M - E$, which can be seen as smooth forms with values in L_Z for any divisor Z with support in E.

The vector space $L^{p,q}_\gamma(M, L_Z)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{Z,\gamma}$. Both subspaces $A^{p,q}_{\gamma,E}(M, L_Z)$ and $A^{p,q}_{cpt}(M - E)$ are dense, and the graph closable operator

$$\overline{\partial}_{cpt} : A^{p,q}_{cpt}(M - E) \rightarrow A^{p,q+1}_{cpt}(M - E)$$

has diverse closed extensions $L^{p,q}_\gamma(M, L_Z) \rightarrow L^{p,q+1}_\gamma(M, L_Z)$. We denote by $\overline{\partial}_s$ the minimal closed extension, i.e. the closure of the graph, and by $\overline{\partial}_w$ the maximal closed extension, i.e. the $\overline{\partial}$-operator in the sense of distributions with respect to compact subsets of $M - E$. Let

$$C^{p,q}_{\gamma,s}(M, L_Z) := \mathrm{Dom}\,\overline{\partial}_s\ ,\quad C^{p,q}_{\gamma,w}(M, L_Z) := \mathrm{Dom}\,\overline{\partial}_w$$

be their domains of definition in $L^{p,q}_\gamma(M, L_Z)$.

In the introduction, we denoted by $L^2_{p,q}(X - \operatorname{Sing} X)$ the space of measurable (p,q)-forms on $X - \operatorname{Sing} X$ which are square-integrable with respect to the Hermitian metric h. Since $\gamma = \pi^* h$ we have now

$$L^2_{p,q}(X - \operatorname{Sing} X) \cong L^{p,q}_\gamma(M, L_\emptyset) = L^{p,q}_\gamma(M, \mathbb{C}),$$

and the operators $\overline{\partial}_s$ and $\overline{\partial}_w$ appearing in the introduction can be identified with

$$\overline{\partial}_s : C^{p,q}_{\gamma,s}(M, \mathbb{C}) \;\to\; L^{p,q+1}_\gamma(M, \mathbb{C}),$$
$$\overline{\partial}_w : C^{p,q}_{\gamma,w}(M, \mathbb{C}) \;\to\; L^{p,q+1}_\gamma(M, \mathbb{C}).$$

So, it makes sense to denote the cohomology of the complexes $(C^{p,*}_{\gamma,s}(M, L_Z), \overline{\partial}_s)$ and $(C^{p,*}_{\gamma,w}(M, L_Z), \overline{\partial}_w)$ analogously to [PS1] by

$$H^{p,q}_{\gamma,s}(M - E, L_Z) \quad \text{and} \quad H^{p,q}_{\gamma,w}(M - E, L_Z),$$

because

$$H^{p,q}_{\gamma,s}(M - E, \mathbb{C}) \;\cong\; H^{p,q}_s(X - \operatorname{Sing} X), \tag{23}$$
$$H^{p,q}_{\gamma,w}(M - E, \mathbb{C}) \;\cong\; H^{p,q}_w(X - \operatorname{Sing} X), \tag{24}$$

where the groups on the right-hand side are the Dolbeault cohomology groups from the introduction.

It follows directly from the definition that the complexes $\left(C^{p,*}_{\gamma,w}(M, L_Z), \overline{\partial}_w\right)$ and $\left(C^{p,*}_{\gamma,E}(M, L_Z), \overline{\partial}_w\right)$ are one and the same. Nevertheless, we keep the two notations for the same object in order to stay consistent with the notation in [PS1]. We remark that our definition of $\left(C^{p,*}_{\gamma,w}(M, \mathbb{C}), \overline{\partial}_w\right)$ differs from the definition of Pardon and Stern at first sight, but is in fact equivalent by Proposition 1.1 in [PS1] (which in turn is Proposition 1.2.3 in [H3]).

We will now identify the Hilbert space adjoints $\overline{\partial}^*_s$ and $\overline{\partial}^*_w$ of $\overline{\partial}_s$ and $\overline{\partial}_w$. Let ϑ be the formal adjoint of $\overline{\partial}$ as computed in Lemma 2.2, and denote by ϑ_{cpt} its action on smooth forms compactly supported in $M - E$:

$$\vartheta_{cpt} : A^{p,q-1}_{cpt}(M - E) \to A^{p,q}_{cpt}(M - E).$$

This operator is graph closable as an operator $L^{p,q-1}_\gamma(M, L_Z) \to L^{p,q}_\gamma(M, L_Z)$, and as for the $\overline{\partial}$-operator, we denote by ϑ_s its minimal closed extension, i.e. the closure of the graph, and by ϑ_w the maximal closed extension, that is the ϑ-operator in the sense of distributions with respect to compact subsets of $M - E$. By (22), it follows that

$$\vartheta_s = - *_{-Z,\gamma} \overline{\partial}_s *_{Z,\gamma}$$

and

$$\vartheta_w = - *_{-Z,\gamma} \overline{\partial}_w *_{Z,\gamma}.$$

By definition, $\overline{\partial}_w = \vartheta_{cpt}^*$, and it follows that

$$\overline{\partial}_w^* = \left(\vartheta_{cpt}^*\right)^* = \vartheta_{cpt} = \vartheta_s = -*_{-Z,\gamma}\overline{\partial}_s *_{Z,\gamma}, \tag{25}$$

if we denote by ϑ_{cpt} also the closure of the graph of ϑ_{cpt}. Analogously, $\vartheta_w = \overline{\partial}_{cpt}^*$ implies

$$\overline{\partial}_s^* = \vartheta_w = -*_{-Z,\gamma}\overline{\partial}_w *_{Z,\gamma}. \tag{26}$$

As usually, this realization of the Hilbert space adjoints yields harmonic representation of cohomology classes and fundamental duality relations:

Theorem 2.3. *As before, let (X, h) be a Hermitian compact complex space of pure dimension n,*

$$\pi : (M, E) \to (X, \operatorname{Sing} X)$$

a resolution of singularities and $\gamma = \pi^* h$ the induced positive semidefinite metric on M. Fix p, $0 \le p \le n$, and a divisor Z with support in E, and suppose that $H_{\gamma,w}^{p,q}(M - E, L_Z)$ is finite dimensional for all q, $1 \le q \le n$. Then:

(a) The (unbounded) operator

$$\overline{\partial}_w : C_{\gamma,w}^{p,q}(M, L_Z) \to L_\gamma^{p,q+1}(M, L_Z)$$

and its Hilbert space adjoint $\overline{\partial}_w^* = \vartheta_s$ have closed ranges, and there is an orthogonal decomposition for each q,

$$L_\gamma^{p,q}(M, L_Z) = \operatorname{range}\overline{\partial}_w \oplus \operatorname{range}\overline{\partial}_w^* \oplus \mathcal{H}_{\gamma,w}^{p,q}(L_Z),$$

where

$$\mathcal{H}_{\gamma,w}^{p,q}(L_Z) := \ker\overline{\partial}_w \cap \ker\overline{\partial}_w^*.$$

There is consequently an induced isomorphism

$$\mathcal{H}_{\gamma,w}^{p,q}(L_Z) \xrightarrow{\cong} H_{\gamma,w}^{p,q}(M - E, L_Z).$$

(b) The same holds with s, $L_Z^* = L_{-Z}$ and $n - p$ in place of w, L_Z and p.
(c) There is a non-degenerate pairing

$$\{\cdot, \cdot\} : H_{\gamma,w}^{p,q}(M - E, L_Z) \times H_{\gamma,s}^{n-p,n-q}(M - E, L_Z^*) \to \mathbb{C}$$

given by

$$\{\eta, \psi\} := \int_M \eta \wedge \psi.$$

Proof. For the proof, we refer to Proposition 1.3 in [PS1] and the Appendix in [KK]. The fact that we consider forms with values in a holomorphic line bundle does not cause additional difficulties. We only mention that the proof depends on (25) and (26), and on the isomorphism

$$*_{Z,\gamma} : \mathcal{H}_{\gamma,w}^{p,q}(L_Z) \xrightarrow{\cong} \mathcal{H}_{\gamma,s}^{n-p,n-q}(L_Z^*). \tag{27}$$

$\square$

As a direct consequence, we obtain the smoothing of cohomology:

Corollary 2.4. *Under the assumptions of Theorem 2.3, the inclusion of complexes*

$$\left(A_{\gamma,E}^{p,*}(M,L_Z),\overline{\partial}\right) \hookrightarrow \left(C_{\gamma,w}^{p,*}(M,L_Z),\overline{\partial}_w\right)$$

induces isomorphisms

$$H^q\left(A_{\gamma,E}^{p,*}(M,L_Z)\right) \xrightarrow{\cong} H_{\gamma,w}^{p,q}(M-E,L_Z), \quad q \geq 0. \tag{28}$$

Proof. Again, just copy the proof of Corollary 1.4 in [PS1]: each element of $\mathcal{H}_{\gamma,w}^{p,q}(L_Z)$ is weakly harmonic on $M-E$, hence smooth on $M-E$ by elliptic regularity so that (28) is surjective. If a smooth form η is weakly exact, one can use the decomposition in (2.3)(a) and elliptic regularity to see that it is smoothly exact. More precisely, the assumption implies $\eta \in \left(\mathcal{H}_{\gamma,w}^{p,q}(L_D)\right)^\perp$, and we can take the canonical solution of the $\overline{\partial}$-equation $\overline{\partial}\psi = \eta$. $\qquad\qquad\square$

Let σ be any positive definite Hermitian metric on M. By (14), (13) and (12), there are natural inclusions of complexes of sheaves

$$\left(\mathcal{C}_{\sigma,E}^{0,*}(L_Z),\overline{\partial}_w\right) \hookrightarrow \left(\mathcal{C}_{\gamma,E}^{0,*}(L_Z),\overline{\partial}_w\right), \tag{29}$$

$$\left(\mathcal{C}_{\gamma,E}^{n,*}(L_Z^*),\overline{\partial}_w\right) \hookrightarrow \left(\mathcal{C}_{\sigma,E}^{n,*}(L_Z^*),\overline{\partial}_w\right), \tag{30}$$

and inclusions of complexes

$$\left(C_{\sigma,B}^{0,*}(M,L_Z),\overline{\partial}_B\right) \hookrightarrow \left(C_{\gamma,B}^{0,*}(M,L_Z),\overline{\partial}_B\right), \tag{31}$$

$$\left(C_{\gamma,B}^{n,*}(M,L_Z^*),\overline{\partial}_B\right) \hookrightarrow \left(C_{\sigma,B}^{n,*}(M,L_Z^*),\overline{\partial}_B\right), \tag{32}$$

for both choices $B=s$ and $B=w$. (30) and (32) hold as well with L_Z instead of L_Z^*. We have chosen the dual bundle to make the presentation consistent with:

Corollary 2.5. *Under the assumptions of Theorem 2.3, the inclusions* (31) *for $B=s$ and* (32) *for $B=w$, or* (31) *for $B=w$ and* (32) *for $B=s$ induce maps in cohomology that are dual with respect to the pairings of Theorem 2.3(c).*

In view of (23), the main part of the proof of Theorem 1.5 is to show that the inclusion (31) for $B=s$ induces an isomorphism

$$H_{\sigma,s}^{0,*}(M-E,L_Z) \xrightarrow{\cong} H_{\gamma,s}^{0,*}(M-E,L_Z) \tag{33}$$

in case $Z = \emptyset$.[12] Recall that

$$H_{\gamma,s}^{0,q}(M-E,L_\emptyset) = H_{\gamma,s}^{0,q}(M-E,\mathbb{C}) \cong H_s^{0,q}(X-\operatorname{Sing} X).$$

Then, it only remains to observe that the complex $(C_{\sigma,s}^{0,*}(M,L_Z),\overline{\partial}_s)$ is the same as $(C_\sigma^{0,*}(M,L_Z),\overline{\partial})$, the standard complex of $L_\sigma^2(Z)$-forms with $\overline{\partial}$ in the sense of distributions (with respect to the positive definite metric σ) which is well known to have the cohomology $H^{0,q}(M,L_Z)$. But that is just Proposition 1.17 in [PS1]:

---

[12]In the next section, we will prove (33) in the more general situation of a line bundle Z fulfilling some kind of semi-negativity condition.

Lemma 2.6. *Let σ be a positive definite Hermitian metric on the complex manifold M, and E a divisor with normal crossings. Then there are equalities of complexes induced by natural inclusions:*

$$\left(C^{p,*}_{\sigma,s}(M, L_Z), \overline{\partial}_s\right) \; \overset{=}{\longrightarrow} \; \left(C^{p,*}_{\sigma,w}(M, L_Z), \overline{\partial}_w\right) = \left(C^{p,*}_{\sigma,E}(M, L_Z), \overline{\partial}\right)$$

$$\overset{=}{\longleftarrow} \; \left(C^{p,*}_\sigma(M, L_Z), \overline{\partial}_w\right) \quad , 0 \le p \le n.$$

The last equality holds without the assumption of normal crossings.

Proof. The proof is given in Propositions 1.12 and 1.17 in [PS1], but a few remarks seem to be useful. The first equality means that the strong and the weak extension of the $\overline{\partial}$-operator on $M - E$ coincide if E is a divisor with normal crossings in a complex manifold M. Since $\operatorname{Dom}\overline{\partial}_s \subset \operatorname{Dom}\overline{\partial}_w$ by definition, one has to show the following: Let $\phi \in L^{p,q}_\sigma(M, L_Z)$ be an L^2_σ-form with values in L_Z such that $\overline{\partial}_w\phi = 0$, i.e. ϕ is $\overline{\partial}$-closed in the sense of distributions on $M - E$. Then there exists a sequence of smooth compactly supported forms $(\phi_j)_j \subset A^{p,q}_{cpt}(M - E)$ such that $\phi_j \to \phi$ in $L^{p,q}_\sigma(L_Z)$ and $\overline{\partial}\phi_j \to \overline{\partial}_w\phi$ in $L^{p,q+1}_\sigma(L_Z)$.

$C^{p,*}_{\sigma,w}$ and $C^{p,*}_{\sigma,E}$ are just two different notations for one and the same object. The last equality is a well-known L^2-extension theorem for the $\overline{\partial}$-operator (see Lemma 2.7 below, which in fact is true for any differential operator of order one). Here, we do not need the assumption of normal crossings. $\square$

Lemma 2.7. *Let A be an analytic subset of an open set $U \subset \mathbb{C}^n$ of codimension $k > 0$. Furthermore, let $f \in L^{2k/2k-1}_{(p,q),loc}(U)$ and $g \in L^1_{(p,q+1),loc}(U)$ be differential forms such that $\overline{\partial}f = g$ in the sense of distributions on $U - A$. Then it follows that $\overline{\partial}f = g$ in the sense of distributions on the whole set U.*

A proof can be found in [R2], Theorem 3.2. Since the first equality in Lemma 2.6 is shown only for a divisor with normal crossings, it doesn't seem appropriate to use it for the proof of Theorem 1.3 and Theorem 1.5 which we claim are true for any resolution of singularities. Actually, the first equality in Lemma 2.6 is not needed, we have just included it for the sake of completeness.

In fact, since M is compact, the groups $H^q(M, \mathcal{K}_M \otimes \mathcal{O}(-Z)) \cong H^{n,q}_\sigma(M, L^*_Z)$ are finite dimensional for $0 \le q \le n$. On the other hand, the last equality in Lemma 2.6 yields

$$H^{n,q}_{\sigma,w}(M - E, L^*_Z) \cong H^{n,q}_\sigma(M, L^*_Z) \; , \; 0 \le q \le n.$$

Thus, Theorem 2.3 implies by duality (using finite-dimensionality of the groups under consideration) that

$$H^{0,q}_{\sigma,s}(M - E, L_Z) \cong H^q(M, \mathcal{O}(Z)) \; , \; 0 \le q \le n.$$

Now, assume that (33) is true for $Z = \emptyset$, namely that

$$H^{0,q}_{\sigma,s}(M - E, \mathbb{C}) \cong H^{0,q}_{\gamma,s}(M - E, \mathbb{C}) \cong H^{0,q}_s(X - \operatorname{Sing} X).$$

This would prove Theorem 1.5. So, we need to investigate (33).

3. PROOF OF THEOREM 1.5

As in section 2, let (X, h) be a Hermitian compact complex space of pure dimension n,

$$\pi : M \to X$$

a resolution of singularities (not necessarily with only normal crossings), and $E = |\pi^{-1}(\mathrm{Sing}\, X)|$ the exceptional set of the desingularization. Let $\gamma := \pi^* h$ be the induced positive semidefinite pseudometric, and σ any Hermitian metric on M. For the proof of Theorem 1.5, we sketch the proof of Theorem 1.3 as it is given in [PS1] and replace the application of the Grauert-Riemenschneider vanishing theorem [GR] by Takegoshi's relative version [T].

In view of the remarks at the end of section 2, it is enough to show that the natural inclusion

$$(C_{\gamma,w}^{n,*}(M,\mathbb{C}), \overline{\partial}_w) \; \hookrightarrow \; (C_{\sigma,w}^{n,*}(M,\mathbb{C}), \overline{\partial}_w) \tag{34}$$

gives an isomorphism

$$H_{\gamma,w}^{n,*}(M - E, \mathbb{C}) \xrightarrow{\;\cong\;} H_{\sigma,w}^{n,*}(M - E, \mathbb{C}).$$

This can be done by showing that the two complexes of sheaves

$$\left(\pi_* \mathcal{C}_{\gamma,E}^{n,*}, \pi_* \overline{\partial}_w\right) \tag{35}$$

and

$$\left(\pi_* \mathcal{C}_{\sigma,E}^{n,*}, \pi_* \overline{\partial}_w\right) \tag{36}$$

both are fine resolutions of $\pi_* \mathcal{K}_M$, where $\mathcal{K}_M$ is the canonical sheaf of M. Since $\overline{\partial}_w$ is the $\overline{\partial}$-operator in the sense of distributions with respect to compact subsets of $M - E$, it follows that $\pi_* \overline{\partial}_w$ is just the $\overline{\partial}$-operator in the sense of distributions on $X - \mathrm{Sing}\, X$, which we will simply denote by $\overline{\partial}$ when the context is clear.

Recall from Lemma 2.1 that $\mathcal{L}_\gamma^{n,0} = \mathcal{L}_\sigma^{n,0}$ because $(n,0)$-forms behave well under a resolution of singularities. Now then, the sheaf cohomology group

$$\mathcal{H}^0(\mathcal{C}_{\gamma,E}^{n,*}) = \mathcal{H}^0(\mathcal{C}_{\sigma,E}^{n,*}) \tag{37}$$

is just the sheaf of germs of square integrable $(n,0)$-forms which are $\overline{\partial}_w$-closed, i.e. which are $\overline{\partial}$-closed in the sense of distributions on $M - E$. Lemma 2.7 implies that such forms are in fact $\overline{\partial}$-closed on M, hence holomorphic on M. So, the sheaf in (37) is nothing else but $\mathcal{K}_M$, the canonical sheaf on M, and both complexes of sheaves $(\mathcal{C}_{\gamma,E}^{n,*}, \overline{\partial}_w)$ and $(\mathcal{C}_{\sigma,E}^{n,*}, \overline{\partial}_w)$ give a resolution of $\mathcal{K}_M$. If the two resolutions were both fine, we wouldn't have much to do. But, whereas the sheaves $\mathcal{C}_{\sigma,E}^{n,*}$ are fine, the sheaves $\mathcal{C}_{\gamma,E}^{n,*}$ are not. This problem can be overcome by considering the direct image complexes (35) and (36). It is clear that the sheaves $\pi_* \mathcal{C}_{\sigma,E}^{n,*}$ are fine because the sheaves $\mathcal{C}_{\sigma,E}^{n,*}$ are already fine. On the other hand, it is not hard to

see that the sheaves $\pi_* \mathcal{C}^{n,*}_{\gamma,E}$ are fine on X ([PS1], Proposition 2.1). Moreover, it follows from our considerations above that

$$\mathcal{H}^0(\pi_* \mathcal{C}^{n,*}_{\gamma,E}) = \mathcal{H}^0(\pi_* \mathcal{C}^{n,*}_{\sigma,E}) = \pi_* \mathcal{K}_M.$$

It remains to show that the two complexes (35) and (36) are exact. However, since $(\mathcal{C}^{n,*}_{\sigma,E}, \overline{\partial}_w)$ is a fine resolution of $\mathcal{K}_M$, the sheaf cohomology groups $\mathcal{H}^q(\pi_* \mathcal{C}^{n,*}_{\sigma,E})$ equal the higher direct image sheaves $R^q \pi_* \mathcal{K}_M$ of $\mathcal{K}_M$. But $R^q \pi_* \mathcal{K}_M = 0$ for $q \geq 1$ by Takegoshi's generalization [T] of the Grauert-Riemenschneider vanishing theorem [GR]. We conclude:

Lemma 3.1. *The complex $(\pi_* \mathcal{C}^{n,*}_{\sigma,E}, \pi_* \overline{\partial}_w)$ is a fine resolution of $\pi_* \mathcal{K}_M$.*

Exactness for the second complex $(\pi_* \mathcal{C}^{n,*}_{\gamma,E}, \overline{\partial}_w)$ follows from a local vanishing result first proved by Ohsawa in case X has only isolated singularities [O1]. The idea, which appears with essentially the same argument in an earlier paper of Demailly [D1], is to approximate the incomplete metric on $X - \operatorname{Sing} X$ by a sequence of complete metrics for which one already knows the local vanishing result by a theorem of Donelly and Fefferman [DF]:

Lemma 3.2. (Donelly-Fefferman [DF]**)** *Let N be a complete Kähler manifold of dimension n, whose Kähler metric ω is given by a potential function $F : N \to \mathbb{R}$ as $\omega = i\partial\overline{\partial}F$ such that $\langle \partial F, \partial F \rangle$ is bounded. Then the L^2-$\overline{\partial}$-cohomology with respect to ω, $H^{p,q}_{(2)}(N, \omega) = 0$ for $p + q \neq n$. In fact, if $\langle \partial F, \partial F \rangle \leq B^2$, and ϕ is a $\overline{\partial}$-closed (p,q)-form on N, $q > 0$ and $p + q \neq n$, then there is a $(p, q-1)$-form ν such that $\overline{\partial}\nu = \phi$ and $\|\nu\| \leq 4B\|\phi\|$.*

Ohsawa gives an elegant proof of Lemma 3.2 using Kähler identities and resulting a priori estimates for the $\overline{\partial}$-operator (see [O1], Theorem 1.1). We will use Ohsawa's idea to prove Lemma 3.2 for forms with values in a semi-positive holomorphic line bundle in the next section (Lemma 4.3).

So, what we do need now are locally complete Kähler metrics on $X - \operatorname{Sing} X$. We only need to consider the $\overline{\partial}$-equation at singular points of X. So, assume that a neighborhood of a point $p \in \operatorname{Sing} X$ is embedded holomorphically into $\mathbb{C}^L$, $L \gg n$, such that $p = 0 \in \mathbb{C}^L$, and let B_c be a ball of very small radius $c > 0$ centered at the origin such that $B_c \cap \operatorname{Sing} X$ is given as the common zero set of holomorphic functions $\{f_1, ..., f_m\}$ in B_c. Following Pardon and Stern, we set

$$F = -\log(c^2 - |z|^2) \tag{38}$$

and

$$F_k = -\log(c^2 - |z|^2) - \frac{1}{k}\log\big(-\log\sum|f_j|^2\big), \tag{39}$$

for $z \in B_c$ and $k > 1$, where c is so small that $\sum |f_j|^2 \ll 1$ on B_c. Let $U := X \cap B_c$. Then some computations yield (see [PS1], Lemma 2.4):

Lemma 3.3. *The metric $\omega_k := i\partial\overline{\partial}F_k$ on $U - U \cap \mathrm{Sing}\,X$ is complete and decreases monotonically to $\omega := i\partial\overline{\partial}F$, pointwise on $U - U \cap \mathrm{Sing}\,X$. $\langle\partial F_k, \partial F_k\rangle_{\omega_k}$ is bounded, independently of k, where $\langle\cdot,\cdot\rangle_{\omega_k}$ denotes the pointwise metric on 1-forms with respect to ω_k.*

So, $U - U \cap \mathrm{Sing}\,X$ carries a sequence of complete metrics ω_k, decreasing to the incomplete metric ω, and we know by Lemma 3.2 that we can solve the $\overline{\partial}$-equation in the L^2-sense with respect to ω_k with a bound that does not depend on k. It is now essential to realize that (n,q)-forms do not only behave well under a resolution of singularities, but also under such a decrease of the metric:

Lemma 3.4. *Let N be a complex manifold of dimension n with a decreasing sequence of Hermitian metrics ω_k, $k \geq 1$, which converges pointwise to a Hermitian metric ω. If $H^{n,q}_{(2)}(N,\omega_k)$ vanishes with an estimate that is independent of k, then $H^{n,q}_{(2)}(N,\omega)$ vanishes with the same estimate.*

For the proof, we refer either to [D1], Theorem 4.1, to [O1], Proposition 4.1, or to [PS1], Lemma 2.3. Combining Lemma 3.2, Lemma 3.3 and Lemma 3.4, we see that the $\overline{\partial}$-equation (in the sense of distributions) can be solved in the L^2-sense for (n,q)-forms, $q \geq 1$, on $U - U \cap \mathrm{Sing}\,X$ with respect to the metric $\omega = i\partial\overline{\partial}F$:

$$H^{n,q}_{(2)}(U - U \cap \mathrm{Sing}\,X, \omega) = 0 \quad, \quad q \geq 1.$$

But ω is quasi-isometric to our original metric h on any smaller subset $U' \subset\subset U$. Hence, we see that in degree (n,q), $q \geq 1$, the $\overline{\partial}$-equation can be solved in the sense of distributions in the L^2-category locally on (the regular set of) a Hermitian complex space.[13] In other words:

Lemma 3.5. *The complex $(\pi_* \mathcal{C}^{n,*}_{\gamma,E}, \pi_* \overline{\partial}_w)$ is a fine resolution of $\pi_* \mathcal{K}_M$.*

That completes the proof of Theorem 1.5. In fact, since $C^{n,q}_{\sigma,w}(M) = C^{n,q}_{\sigma,E}(M)$, Lemma 3.1 yields

$$H^{n,q}_{\sigma,w}(M - E, \mathbb{C}) = H^q(\Gamma(X, \pi_* \mathcal{C}^{n,*}_{\sigma,E})) \cong H^q(X, \pi_* \mathcal{K}_M)$$

and Lemma 3.5 gives

$$H^{n,q}_{\gamma,w}(M - E, \mathbb{C}) = H^q(\Gamma(X, \pi_* \mathcal{C}^{n,*}_{\gamma,E})) \cong H^q(X, \pi_* \mathcal{K}_M).$$

In order to compare $\overline{\partial}_w$- and $\overline{\partial}_s$-cohomology later on (Theorem 6.4), we need a more general version of Theorem 1.5 which follows from slightly more general versions of Lemma 3.1 and Lemma 3.5. We will basically consider the following situation: Let Z be a divisor with support in E and L_Z the associated line bundle such that $\mathcal{O}(L_Z) \cong \mathcal{O}(Z)$, and assume that L_Z is semi-positive. Then the two complexes $(\pi_* \mathcal{C}^{n,*}_{\gamma,E}(L_Z), \overline{\partial}_w)$ and $(\pi_* \mathcal{C}^{n,*}_{\sigma,E}(L_Z), \overline{\partial}_w)$ both are fine resolutions of $\pi_*\big(\mathcal{K}_M \otimes \mathcal{O}(Z)\big)$. A similar reasoning leads to a version of Theorem 1.5 with values in certain holomorphic line bundles (Theorem 4.1).

---

[13] See also Remark 15.16 in Demailly's introduction to Hodge theory [D2].

4. Generalization of Theorem 1.5

Again, let (X, h) be a Hermitian compact complex space of pure dimension n, $\pi : M \to X$ any resolution of singularities, and $Z := \pi^{-1}(\operatorname{Sing} X)$ the unreduced exceptional divisor. Again, we denote by $E = |\pi^{-1}(\operatorname{Sing} X)| = |Z|$ the exceptional set of the desingularization, by $\gamma := \pi^* h$ the induced positive semidefinite pseudometric, and by σ any Hermitian metric on M. Let $L_{Z-|Z|}$ be the holomorphic line bundle associated to the divisor $Z - |Z|$ such that $\mathcal{O}(L_{Z-|Z|}) \cong \mathcal{O}(Z - |Z|)$ with the Hermitian metric (20). For simplicity, we denote the restriction of $L_{Z-|Z|}$ to $M - E$ by $L_{Z-|Z|}$, as well. Let $\pi_* L_{Z-|Z|}$ be the push-forward of (the restriction of) $L_{Z-|Z|}$ to $X - \operatorname{Sing} X$. We use the same constructions and notations for the dual bundle $L_{Z-|Z|}^* \cong L_{|Z|-Z}$.

In this section, we will show that Theorem 1.5 remains valid for forms with values in $L_{Z-|Z|}$ under the assumption that $L_{Z-|Z|}$ is locally semi-negative with respect to the base space X, i.e. if each point $x \in X$ has an open neighborhood U_x such that $L_{Z-|Z|}$ is semi-negative on $\pi^{-1}(U_x)$.

Theorem 4.1. *Let X be a Hermitian compact complex space of pure dimension, $\pi : M \to X$ any resolution of singularities, and $0 \leq q \leq n = \dim X$. If $L_{Z-|Z|}$ is locally semi-negative with respect to X, then*

$$H_s^{0,q}(X - \operatorname{Sing} X, \pi_* L_{Z-|Z|}) \cong H^q(M, \mathcal{O}(Z - |Z|)), \qquad (40)$$

where Z is the unreduced exceptional divisor $Z = \pi^{-1}(\operatorname{Sing} X)$ and $\pi_* L_{Z-|Z|}$ the push-forward of the line bundle associated to the divisor $Z - |Z|$.

Note that

$$H_s^{0,q}(X - \operatorname{Sing} X, \pi_* L_{Z-|Z|}) \cong H_{\gamma,s}^{0,q}(M - E, L_{Z-|Z|})$$

by definition. So, by duality (Theorem 2.3), (40) is equivalent to

$$H_{\gamma,w}^{n,q}(M - E, L_{|Z|-Z}) \cong H^q(M, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)),$$

and that is what we are going to prove in the following. As in the proof of Theorem 1.5, that can be done by showing that

$$H_{\gamma,w}^{n,q}(M - E, L_{|Z|-Z}) \cong H_{\sigma,w}^{n,q}(M - E, L_{|Z|-Z}).$$

So, we have to show that Lemma 3.1 and Lemma 3.5 are valid for the complexes of sheaves of germs of forms with values in $L_{|Z|-Z}$, i.e. we have to show that the two complexes of sheaves

$$\left(\pi_* \mathcal{C}_{\gamma,E}^{n,*}(L_{|Z|-Z}), \pi_* \overline{\partial}_w \right) \qquad (41)$$

and

$$\left(\pi_* \mathcal{C}_{\sigma,E}^{n,*}(L_{|Z|-Z}), \pi_* \overline{\partial}_w \right) \qquad (42)$$

both are fine resolutions of $\pi_* \left(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z) \right)$ under the assumption that $L_{|Z|-Z}$ is locally semi-positive with respect to X. Fineness and exactness at $q = 0$ follow as in the proof of Theorem 1.5.

For the generalization of Lemma 3.1, we observe that Takegoshi's vanishing theorem is true for forms with values in a semi-positive holomorphic line bundle (see [T], Remark 2(a)). But Takegoshi's theorem is a statement that is local with respect to the base space X. Hence, we get immediately:

Lemma 4.2. *The complex of sheaves* $(\pi_* \mathcal{C}_{\sigma,E}^{n,*}(L_{|Z|-Z}), \pi_* \overline{\partial}_w)$ *is a fine resolution of* $\pi_* \big(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z) \big)$ *if* $L_{|Z|-Z}$ *is locally semi-positive with respect to* X.

Now, for the generalization of Lemma 3.5, we observe that Lemma 3.2 and Lemma 3.4 remain valid for (n,q)-forms with values in a semi-positive Hermitian line bundle on $X - \operatorname{Sing} X$. That is clear for Lemma 3.4, but it seems appropriate to check it carefully in the case of Lemma 3.2. Following closely Ohsawa's proof in [O1], Theorem 1.1, we get:

Lemma 4.3. *Let N be a complete Kähler manifold of dimension n, whose Kähler metric ω is given by a potential function $F : N \to \mathbb{R}$ as $\omega = i\partial\overline{\partial}F$ such that $\langle \partial F, \partial F \rangle_\omega$ is bounded, and let (V, H) be a semi-positive Hermitian line bundle on N. Then the L^2-$\overline{\partial}$-cohomology in the sense of distributions with respect to ω for forms with values in V, $H_{(2)}^{n,q}(N, V, \omega) = 0$ for $q > 0$. In fact, if $\langle \partial F, \partial F \rangle_\omega \leq B^2$, and ϕ is a $\overline{\partial}$-closed (n,q)-form on N, $q > 0$, then there is a $(n, q-1)$-form ν such that $\overline{\partial}\nu = \phi$ and $\|\nu\|_{\omega,H} \leq 4B\|\phi\|_{\omega,H}$.*

Proof. Let $D = D' + D'' = D' + \overline{\partial}$ be the Chern connection on V. It follows by [H4], Lemma 4.1.1, that it is enough to show that

$$\|u\|_{\omega,H} \leq 4|\partial F|_{\infty,\omega} \|(D'')^* u\|_{\omega,H} \tag{43}$$

for any $u \in \ker D'' \cap \operatorname{Dom}(D'')^* \cap L_{(2)}^{n,q}(V, H, \omega)$ if $q \geq 1$. The estimate (43) can be proved as follows:

For any differential form η, let $e(\eta)$ denote left multiplication by η, $[,]$ the commutator with appropriate weight (i.e. $[S, T] = S \circ T - (-1)^{\deg S \deg T} T \circ S$), and $*$ the L^2-adjoint. With $\Lambda := e(i\partial\overline{\partial}F)^*$, it is well-known that $[D'', \Lambda] = i(D')^*$ and $[e(\partial F), \Lambda] = ie(\overline{\partial}F)^*$ (see [D2], 13.1, and [W], V.(3.22)). Note that Λ, $e(\partial F)^*$ and $e(\overline{\partial}F)^*$ are independent of the Hermitian vector bundle (V, H). Therefore:

$$
\begin{aligned}
[D'', e(\overline{\partial}F)^*] &= D'' \circ e(\overline{\partial}F)^* + e(\overline{\partial}F)^* \circ D'' \\
&= -iD'' \circ [e(\partial F), \Lambda] - i[e(\partial F), \Lambda] \circ D'' \\
&= [e(i\partial\overline{\partial}F), \Lambda] + ie(\partial F) \circ [D'', \Lambda] + i[D'', \Lambda] \circ e(\partial F) \\
&= [e(i\partial\overline{\partial}F), \Lambda] - [e(\partial F), (D')^*].
\end{aligned}
$$

Reorganzing this, we have

$$[e(i\partial\overline{\partial}F), \Lambda] = [D'', e(\overline{\partial}F)^*] + [e(\partial F), (D')^*].$$

For a compactly supported smooth form u, that yields:

$$
\begin{aligned}
\left|\left([e(i\partial\overline{\partial}F,\Lambda]u,u\right)_{\omega,H}\right| &= \left|(u,e(\overline{\partial}F)\circ(D'')^*u)_{\omega,H} + (D''u,e(\overline{\partial}F)u)_{\omega,H}\right. \\
&\quad + \left.(e(\partial F)\circ(D')^*u,u)_{\omega,H} + (e(\partial F)u,D'u)_{\omega,H}\right| \\
&\leq |\partial F|_{\infty,\omega}\|u\|_{\omega,H} \\
&\quad \cdot\left(\|(D'')^*u\|_{\omega,H} + \|D''u\|_{\omega,H} + \|(D')^*u\|_{\omega,H} + \|D'u\|_{\omega,H}\right) \\
&\leq 4|\partial F|_{\infty,\omega}\|u\|_{\omega,H}\left(\|(D'')^*u\|_{\omega,H} + \|D''u\|_{\omega,H}\right),
\end{aligned}
$$

where the last step follows from the Bochner-Kodaira-Nakano identity

$$
\begin{aligned}
\|(D')^*u\|^2_{\omega,H} + \|D'u\|^2_{\omega,H} &= \|(D'')^*u\|^2_{\omega,H} + \|D''u\|^2_{\omega,H} \\
&\quad - \int_N \langle[i\Theta(V),\Lambda]u,u\rangle_{\omega,H}dV_\omega
\end{aligned}
$$

(see [D2], Theorem 13.12) and the fact that

$$\langle[i\Theta(V),\Lambda]u,u\rangle_{\omega,H} \geq 0$$

because V is semi-positive (see [D2], 13.6). In this step one sees that the Lemma is also valid for $(0,q)$-forms with values in a semi-negative line bundle. On the other hand,

$$([e(i\partial\overline{\partial}F,\Lambda]u,u)_{\omega,H} = (p+q-n)\|u\|^2_{\omega,H}$$

if u is a (p,q)-form (see [W], Proposition V.1.1(c)). So, the estimate (43) is valid for smooth (n,q)-forms with compact support if $q \geq 1$. As (N,ω) is complete, that implies (43) and the lemma is proved (cf. [D2], Proposition 12.2). $\qquad\square$

The rest of the proof of Lemma 3.5 goes through as before and we obtain:

Lemma 4.4. *The complex of sheaves $(\pi_*\mathcal{C}^{n,*}_{\gamma,E}(L_{|Z|-Z}),\pi_*\overline{\partial}_w)$ is a fine resolution of $\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z))$ if $L_{|Z|-Z}$ is locally semi-positive with respect to X.*

That completes the proof of Theorem 4.1.

A few words about the assumption on $L_{|Z|-Z}$ are in order. In any case, at least L_{-Z} is locally semi-positive with respect to the base space X. We have to check the following: Let $p \in X$ be a singular point. Then there exists a neighborhood U_p of p such that L_{-Z} is semi-positive on $\pi^{-1}(U_p)$. For that, we follow Demailly [D2], Example 11.22.

As in the previous section, assume that a neighborhood of $p \in \operatorname{Sing} X$ is embedded holomorphically into $\mathbb{C}^L$, $L \gg n$, such that $p = 0 \in \mathbb{C}^L$, and let B_c be a ball of very small radius $c > 0$ centered at the origin such that $B_c \cap \operatorname{Sing} X$ is given as the common zero set of holomorphic functions $\{f_1,...,f_m\}$ in B_c. Let

$$\Phi := \sum |f_j|^2 \ll 1.$$

In $\pi^{-1}(B_c \cap X)$, the divisor $Z = \pi^{-1}(\operatorname{Sing} X)$ is given as the common zero set (counted with multiplicities) of the holomorphic functions $\{\pi^*f_1,...,\pi^*f_m\}$. So,

$\pi^*\sqrt{\Phi}$ gives a non-vanishing section in $L_{-Z} \cong L_Z^*$. Hence, $\pi^*\Phi$ gives a Hermitian metric on L_Z by setting

$$\langle \alpha, \beta \rangle_{L_Z} = (\alpha \cdot \pi^*\sqrt{\Phi})(\beta \cdot \pi^*\sqrt{\Phi})$$

which is isomorphic to our original metric (20). Let us now consider the dual metric on L_{-Z}. If τ is a local trivialization of L_{-Z}, then this metric gives

$$\|\eta\|_{L_{-Z}}^2 = \frac{|\tau(\eta)|^2}{|\tau(\pi^*f_1)|^2 + \cdots |\tau(\pi^*f_m)|^2}.$$

The associated weight function in this trivialization is

$$\varphi = \log \sum |\tau(\pi^*f_j)|^2,$$

and so L_{-Z} is a semi-positive holomorphic line bundle on $\pi^{-1}(B_c \cap X)$ since

$$i\partial\overline{\partial}\varphi \geq 0.$$

We mention briefly a second approach to prove semi-positivity of L_{-Z} on $\pi^{-1}(B_c \cap X)$ as above. Any desingularization of $B_c \cap X$ has to start with the blow-up of the origin, say $\pi_1 : N \to B_c$. It is well known that the exceptional divisor $D = \pi_1^{-1}(\{0\})$ of this blow-up has negativ self-intersection so that L_{-D} is a positive bundle. Let $\pi = \pi_1 \circ \pi_2$. Then $L_{-Z} = \pi_2^* L_{-D}$ is semi-positive.

Both considerations suggest that semi-positivity of $L_{|Z|-Z}$ is a reasonable pre-condition which is fulfilled in many situations, e.g. if $\dim X = 1$ or if Z has the same order on each of its irreducible components. The latter situation appears for example in the important case of conical singularities where the desingularization is given by a single blow-up.

5. L^2-DOLBEAULT COHOMOLOGY AT ISOLATED SINGULARITIES

As a preparation for the proof of Theorem 1.6, we study different kinds of L^2-Dolbeault cohomology in this section. Besides $\overline{\partial}_w$ and $\overline{\partial}_s$, we consider two other closed extensions of the $\overline{\partial}$-operator which we will define in the following.

Throughout this section, let X be a pure n-dimensional complex analytic set in $\mathbb{C}^L$ with an isolated singularity at the origin such that $\operatorname{Reg} X$ carries the restriction of the Euclidean metric. For small $r > 0$, let $X^* = X - \{0\}$, $X_r = X \cap B_r(0)$ and $X_r^* = X_r - \{0\}$. Moreover, let

$$
\begin{aligned}
L_{cpt}^{p,q}(X_r^*) &:= \{f \in L^{p,q}(X_r^*) : \operatorname{supp} f \subset\subset X_r^*\}, \\
L_0^{p,q}(X_r^*) &:= \{f \in L^{p,q}(X_r^*) : \operatorname{supp} f \cap \{0\} = \emptyset\}, \\
L_b^{p,q}(X_r^*) &:= \{f \in L^{p,q}(X_r^*) : \operatorname{supp} f \cap bB_r(0) = \emptyset\},
\end{aligned}
$$

where $\operatorname{supp} f$ is taken in X. We may now consider the operators (defined in the sense of distributions)

$$
\begin{aligned}
\overline{\partial}_{cpt} &: \quad L_{cpt}^{p,q}(X_r^*) \to L_{cpt}^{p,q+1}(X_r^*), \\
\overline{\partial}_0 &: \quad L_0^{p,q}(X_r^*) \to L_0^{p,q+1}(X_r^*), \\
\overline{\partial}_b &: \quad L_b^{p,q}(X_r^*) \to L_b^{p,q+1}(X_r^*),
\end{aligned}
$$

and the formal adjoints $\vartheta_{cpt} = -*\overline{\partial}_{cpt}*$, $\vartheta_0 = -*\overline{\partial}_0*$, $\vartheta_b = -*\overline{\partial}_b*$. All these operators are densely defined and graph closable because the smooth forms with compact support are dense in each of the special L^2-spaces under consideration. On the other hand, each of these L^2-spaces is dense in $L^{p,q}(X_r^*)$ resp. $L^{p,q+1}(X_r^*)$. So, we can now consider the closed extensions of $\overline{\partial}_{cpt}$, $\overline{\partial}_0$, $\overline{\partial}_b$, respectively ϑ_{cpt}, ϑ_0 and ϑ_b as operators

$$
L^{p,q}(X_r^*) \to L^{p,q+1}(X_r^*)
$$

respectively

$$
L^{p,q+1}(X_r^*) \to L^{p,q}(X_r^*).
$$

The maximal closed extensions of all these operators are the $\overline{\partial}$ respectively the ϑ-operator in the sense of distributions $\overline{\partial}_w$ and ϑ_w, but we obtain some new minimal closed extensions. First note that we already discussed the operators

$$
\overline{\partial}_{cpt}^{**} = \overline{\partial}_s \quad \text{and} \quad \vartheta_{cpt}^{**} = \vartheta_s,
$$

coming with boundary conditions at 0 and $bB_r(0) \cap X$.

We denote the other closures of the graphs as follows:

$$
\begin{aligned}
\overline{\partial}_0^{**} &=: \overline{\partial}_{s,w} \quad \text{and} \quad \vartheta_0^{**} =: \vartheta_{s,w}, \\
\overline{\partial}_b^{**} &=: \overline{\partial}_{w,s} \quad \text{and} \quad \vartheta_b^{**} =: \vartheta_{w,s}.
\end{aligned}
$$

Here, the operators $\overline{\partial}_{s,w}$ and $\vartheta_{s,w}$ have boundary conditions at 0, whereas $\overline{\partial}_{w,s}$ and $\vartheta_{w,s}$ come with boundary conditions at $bB_r(0) \cap X$.

For the L^2-adjoints, we obtain:

Lemma 5.1.
$$\overline{\partial}^*_{s,w} = \vartheta_{w,s} \quad and \quad \overline{\partial}^*_{w,s} = \vartheta_{s,w}.$$

Proof. We will prove $\overline{\partial}^*_{s,w} = \vartheta_{w,s}$, the second statement follows similarly. It is enough to show that
$$\mathrm{Dom}\,\overline{\partial}^*_{s,w} = \mathrm{Dom}\,\vartheta_{w,s},$$

for if a form f is in both domains, then it is clear that $\overline{\partial}^*_{s,w}f = \vartheta_w f = \vartheta_{w,s}f$. Let $f \in \mathrm{Dom}\,\vartheta_{w,s}$. So, f can be approximated by forms with support away from the boundary $bB_r(0)$. Hence, partial integration is possible:
$$(f, \overline{\partial}_{s,w}g)_{X_r^*} = (\vartheta_{w,s}f, g)_{X_r^*} \; \forall g \in \mathrm{Dom}\,\overline{\partial}_{s,w},$$

since such g can be approximated by forms with support away from the origin.

Conversely, let $f \in \mathrm{Dom}\,\overline{\partial}^*_{s,w}$. Let $\chi \in C^\infty(X)$ be a cut off function such that $\chi \equiv 0$ in a neighborhood of the origin and $\chi \equiv 1$ outside $B_{r/2}(0) \cap X$. Then
$$\chi f \in \mathrm{Dom}\,\overline{\partial}^*_w = \mathrm{Dom}\,\vartheta_s \subset \mathrm{Dom}\,\vartheta_{w,s}.$$

On the other hand, is is clear that $(1-\chi)f \in L_b^{p,q}(X_r^*)$ is in $\mathrm{Dom}\,\vartheta_{w,s}$. Hence:
$$f = \chi f + (1-\chi)f \; \in \mathrm{Dom}\,\vartheta_{w,s}.$$

$\square$

Let $H_{s,w}^{p,q}(X_r^*)$ and $H_{w,s}^{p,q}(X_r^*)$ be the L^2-Dolbeault cohomology groups with respect to the $\overline{\partial}_{s,w}$ and the $\overline{\partial}_{w,s}$-operator. If we define the spaces of harmonic forms as

$$\begin{aligned}
\mathcal{H}_{s,w}^{p,q}(X_r^*) &= \ker\overline{\partial}_{s,w} \cap \ker\overline{\partial}^*_{s,w}, \\
\mathcal{H}_{w,s}^{p,q}(X_r^*) &= \ker\overline{\partial}_{w,s} \cap \ker\overline{\partial}^*_{w,s},
\end{aligned}$$

then we obtain as in Theorem 2.3 the duality:
$$\mathcal{H}_{s,w}^{p,q}(X_r^*) \cong \mathcal{H}_{w,s}^{n-p,n-q}(X_r^*), \tag{44}$$

where the isomorphism is given by application of the $*$-operator. (44) extends to an isomorphism of the associated cohomology classes if one of these classes is finite-dimensional. So, we can include a dual statement in all the following results.

Let us now review what is known about the different kinds of L^2-Dolbeault cohomology groups. We have seen in the present paper (by the proof of [PS1]):

Lemma 5.2.
$$H_s^{0,n-q}(X_r^*) \cong H_w^{n,q}(X_r^*) \cong \{0\} \;, \; q \geq 1.$$

Proof. Let $f \in L^{n,q}(X_r^*)$ such that $\overline{\partial}_w f = 0$. Then we have seen in section 3 that there exists a local L^2-solution of the $\overline{\partial}_w$-equation at the origin. Since X_r has strongly pseudoconvex boundary in X, this solution can be extended to X_r^* by standard methods (see e.g. section VIII.4 in [LM]). Another proof of the lemma is in [FOV1], Theorem 1.2. $\qquad\square$

We will now show that also

$$H_{w,s}^{0,q}(X_r^*) \cong \{0\} \ , \ 1 \le q \le n-1.$$

This statement can be reduced to another vanishing result of Fornæss, Øvrelid and Vassiliadou, namely Proposition 3.1 in [FOV1]:

Lemma 5.3. *Let $p+q < n$, $q > 0$, and $0 < r_0 < r$. Let $f \in L^{p,q}(X_r^*)$ such that $\overline{\partial}_w f = 0$ and $\operatorname{supp} f \subset X_{r_0}$. Then there exists $u \in L^{p,q-1}(X_r^*)$ such that $\overline{\partial}_w u = f$. If u is extended trivially by 0 to X^*, then $\overline{\partial} u = f$ on X^*.*

Combining Lemma 5.2 with Lemma 5.3, we obtain:

Lemma 5.4. *Let $c > 0$ small enough. Then:*

$$H_{s,w}^{n,n-q}(X_c^*) \cong H_{w,s}^{0,q}(X_c^*) \cong \{0\} \ , \ 1 \le q \le n-1.$$

Proof. Let $f \in L^{0,q}(X_c^*)$ such that $\overline{\partial}_{w,s} f = 0$. Let f^0 be the trivial extension of f to X^*. Then $\overline{\partial}_w f^0 = 0$ on X^* because f can be approximated in the graph norm by a sequence in $L_b^{0,q}(X_c^*)$ and this sequence approximates f^0 as well.

Choose $r > r_0 > c > 0$ adequately, and let u be the solution to

$$\overline{\partial} u = f^0$$

from Lemma 5.3. Now then, let $\chi \in C^\infty(X)$ be a cut-off function such that $\chi \equiv 1$ in a neighborhood of the origin and $\chi \equiv 0$ outside $X \cap B_{c/2}(0)$. Consider

$$f' := f - \overline{\partial}_w(\chi u),$$

which is $\overline{\partial}_w$-closed, vanishes in a neighborhood of the origin and equals f outside the ball $B_{c/2}(0)$, especially close to the boundary $X \cap bB_c(0)$. But then $\overline{\partial}_s f' = 0$ for the $\overline{\partial}_s$ operator on X_c^*. By Lemma 5.2, there exists $g \in L^{0,q-1}(X_c^*)$ such that

$$\overline{\partial}_s g = f'.$$

It is then clear that also $\overline{\partial}_{w,s} g = f'$. On the other hand, χu is identically zero outside $B_{c/2}(0)$, yielding

$$\overline{\partial}_{w,s}(\chi u) = \overline{\partial}_w(\chi u)$$

on X_c^*. Hence $g' := g + \chi u$ is the desired solution to the $\overline{\partial}_{w,s}$-equation:

$$\overline{\partial}_{w,s} g' = \overline{\partial}_{w,s} g + \overline{\partial}_{w,s}(\chi u) = f' + \overline{\partial}_w(\chi u) = f.$$

$\qquad\square$

We will see later that also $H_{s,w}^{n,n}(X_c^*) \cong H_{w,s}^{0,0}(X_c^*) \cong \{0\}$ by another result of Fornæss, Øvrelid and Vassiliadou, Theorem 1.2 in [FOV1].

6. The $\overline{\partial}_s$-complex

Assume from now on that X has only isolated singularities and that $\pi : M \to X$ is a resolution of singularities with only normal crossings as before, an that $L_{|Z|-Z}$ is locally semi-positive with respect to the base space X.

In this section, we will prove Theorem 1.6 by a strategy similar to the proof of Theorem 1.5 and Theorem 4.1, namely by comparing two fine resolutions of

$$\pi_* \left(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z) \right). \tag{45}$$

As in section 4, the first fine resolution of (45) is given by the direct image complex

$$(\pi_* \mathcal{C}^{n,*}_{\sigma,E}(L_{|Z|-Z}), \pi_* \overline{\partial}_w) \tag{46}$$

which is exact by Lemma 4.2 relying on Takegoshi's vanishing theorem. For the second resolution, we need to define a $\overline{\partial}_s$-complex.

Hence, we have to localize the $\overline{\partial}_s$-operator (which has been defined before only for L^2-forms) appropriately. For an open set $U \subset M$, let $A^{p,q}_{c,E}(U)$ be the set of smooth (p,q)-forms on U that have support away from E. For

$$f \in \Gamma(U, \mathcal{L}^{p,q}_\gamma) = L^{p,q}_{\gamma,loc}(U),$$

we say that $f \in \operatorname{Dom} \overline{\partial}_{s,loc}$ if $\overline{\partial}_w f \in \Gamma(U, \mathcal{L}^{p,q+1}_\gamma)$ and there exists a sequence of smooth forms with support away from the exceptional set E, $\{\phi_j\} \subset A^{p,q}_{c,E}(U)$, such that

$$\phi_j \ \to \ f \ \text{in } L^{p,q}_\gamma(K), \tag{47}$$
$$\overline{\partial}\phi_j \ \to \ \overline{\partial}_w f \ \text{in } L^{p,q+1}_\gamma(K), \tag{48}$$

on each compact subset $K \subset U$ as $j \to \infty$. In that case we set $\overline{\partial}_{s,loc} f = \overline{\partial}_w f$.

We should compare this definition with the $\overline{\partial}_s$-operator on L^2-forms. For an open set $U \subset M$, let

$$L^{p,q}_{\gamma,E}(U) := \{f \in L^{p,q}_\gamma(U) : \operatorname{supp} f \cap E = \emptyset\}$$

and

$$\overline{\partial}_E : L^{p,q}_{\gamma,E}(U) \to L^{p,q+1}_{\gamma,E}(U)$$

the $\overline{\partial}$-operator in the sense of distributions on $L^2_{\gamma,E}$-forms which is densely defined and graph closable as an operator $L^{p,q}_\gamma(U) \to L^{p,q+1}_\gamma(U)$. Then we have the L^2_γ-adjoints

$$\vartheta_{w,E} = \overline{\partial}^*_E \ , \ \overline{\partial}_{s,E} = \overline{\partial}^{**}_E,$$

where $\overline{\partial}_{s,E}$ is the closure of the graph. Thus, an L^2_γ-form $f \in L^{p,q}_\gamma(U)$ is in $\operatorname{Dom} \overline{\partial}_{s,E}$ if $\overline{\partial}_w f \in L^{p,q+1}_\gamma(U)$ and there exists a sequence of forms with support

away from the exceptional set E, $\{\phi_j\} \subset L^{p,q}_{\gamma,E}(U)$, such that

$$\phi_j \rightarrow f \text{ in } L^{p,q}_\gamma(U), \tag{49}$$

$$\overline{\partial}\phi_j \rightarrow \overline{\partial}_w f \text{ in } L^{p,q+1}_\gamma(U). \tag{50}$$

Consider $f \in L^{p,q}_\gamma(U) \subset L^{p,q}_{\gamma,loc}(U)$. Assume $f \in \mathrm{Dom}\, \overline{\partial}_{s,E}$. Then it is clear that $f \in \mathrm{Dom}\, \overline{\partial}_{s,loc}$ and $\overline{\partial}_{s,loc}f = \overline{\partial}_{s,E}f = \overline{\partial}_w f$. Just take a sequence as in (49) and (50), multiply it with a suitable cut-off sequence exhausting U, and use a smoothing mollifier. To simplify the notation, we will denote both, $\overline{\partial}_{s,loc}$ and $\overline{\partial}_{s,E}$, by $\overline{\partial}_s$ if it does not cause any confusion.

We have to check that the notion of $\overline{\partial}_{s,loc}$ behaves well with respect to sheaves. Let $V \subset U$ be two open sets, and $f \in \mathrm{Dom}\, \overline{\partial}_{s,loc}(U)$. Then it is clear that $f|_V \in \mathrm{Dom}\, \overline{\partial}_{s,loc}(V)$ and $\overline{\partial}_{s,loc}(V)f|_V = \big(\overline{\partial}_{s,loc}(U)f\big)|_V$.

The only real issue we have to check is the following: Let $V = \cup_\mu V_\mu$ be a union of open sets, $f \in L^{p,q}_{\gamma,loc}(V)$ and $f|_{V_\mu} \in \mathrm{Dom}\, \overline{\partial}_{s,loc}(V_\mu)$ for all μ. We claim that $f \in \mathrm{Dom}\, \overline{\partial}_{s,loc}(V)$.

To see this, we can assume (by taking a refinement if necessary) that the open cover $\mathcal{V} := \{V_\mu\}$ is locally finite, and choose a partition of unity $\{\varphi_\mu\}$ for $\mathcal{V}$. On V_μ choose a sequence $\{f^\mu_j\} \subset A^{p,q}_{c,E}(V_\mu)$ as in (47), (48), and consider

$$f_j := \sum_\mu \varphi_\mu f^\mu_j.$$

It is clear that $\{f_j\} \subset A^{p,q}_{c,E}(V)$. If $K \subset V$ is compact, then $K \cap \mathrm{supp}\, \varphi_\mu$ is a compact subset of V_μ for each μ, so that $\{f^\mu_j\}$ and $\{\overline{\partial}f^\mu_j\}$ converge in the L^2_γ-sense to f resp. $\overline{\partial}_w f$ on $K \cap \mathrm{supp}\, \varphi_\mu$. But then $\{f_j\}$ and $\{\overline{\partial}f_j\}$ converge in the L^2_γ-sense to f resp. $\overline{\partial}_w f$ on K and that was to show.

So, we obtain for each $p \geq 0$ the $\overline{\partial}_s$-complex $(\mathcal{F}^{p,*}_{\gamma,E}, \overline{\partial}_s)$ where

$$\mathcal{F}^{p,q}_{\gamma,E} := \mathcal{L}^{p,q}_\gamma \cap \overline{\partial}^{-1}_{s,loc}\mathcal{L}^{p,q+1}_\gamma$$

is the sheaf given by

$$\mathcal{F}^{p,q}_{\gamma,E}(U) = L^{p,q}_{\gamma,loc}(U) \cap \mathrm{Dom}\, \overline{\partial}_{s,loc}(U).$$

We can now prove Theorem 1.6 by showing that $(\pi_*\mathcal{F}^{n,*}_{\gamma,E}, \pi_*\overline{\partial}_s)$ is another fine resolution of (45), because $\pi_*\overline{\partial}_s$ acts on global sections as the $\overline{\partial}_s$-operator from the introduction:

$$H^q(\Gamma(X, \pi_*\mathcal{F}^{n,*}_{\gamma,E})) = H^{n,q}_s(X - \mathrm{Sing}\, X).$$

To prove exactness, we will start by showing that

$$0 \rightarrow \pi_* \left(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)\right) \hookrightarrow \pi_*\mathcal{F}^{n,0}_{\gamma,E} \xrightarrow{\pi_*\overline{\partial}_s} \pi_*\mathcal{F}^{n,1}_{\gamma,E} \tag{51}$$

is an exact sequence of sheaves.

Let us first show that
$$\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z) \subset \mathcal{F}_{\gamma,E}^{n,0}.$$
That can be done by a simple cut-off procedure that we recall from [PS1]. Let $U \subset M$ be an open set and $\phi \in \Gamma(U, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$. Then $\phi \in L_{\gamma,loc}^{n,0}(U)$, and we have to show that $\phi \in \mathrm{Dom}\, \overline{\partial}_{s,loc}$. We can assume that $U = \pi^{-1}(U')$, where U' is an open neighborhood of an isolated singularity $p \in \mathrm{Sing}\, X$, and that U' is embedded holomorphically in $\mathbb{C}^L$ such that $p = 0 \in \mathbb{C}^L$.

As in [PS1], Lemma 3.6, let $\rho_k : \mathbb{R} \to [0,1]$, $k \geq 1$, be smooth cut-off functions satisfying
$$\rho_k(x) = \left\{ \begin{array}{ll} 1 & , \ x \leq k, \\ 0 & , \ x \geq k+1, \end{array} \right.$$
and $|\rho_k'| \leq 2$. Moreover, let $r : \mathbb{R} \to [0, 1/2]$ be a smooth increasing function such that
$$r(x) = \left\{ \begin{array}{ll} x & , \ x \leq 1/4, \\ 1/2 & , \ x \geq 3/4, \end{array} \right.$$
and $|r'| \leq 1$. We do now need a function measuring the distance to the exceptional set E in M. A good choice is just the pull-back of the euclidean distance in $\mathbb{C}^L$. So, let
$$F := \left(\sum_{j=1}^{L} |w_j|^2 \right)^{1/2},$$
where $w_1, ..., w_L$ are the Cartesian coordinates of $\mathbb{C}^L$. Since the metric h is quasi-isometric to the Euclidean metric in $\mathbb{C}^L$, we have $|\overline{\partial}F|_h \lesssim 1$. As cut-off functions we can use
$$\mu_k := \rho_k(\log(-\log r(\pi^* F)))$$
on M. Thus, we claim that
$$\phi_k := \mu_k \phi$$
is a suitable sequence of smooth forms with support away from E. Let $K \subset U$ be a compact subset. It is clear that $\phi_k \to \phi$ in $L_\gamma^{n,0}(K)$ and that $\mu_k \overline{\partial}\phi \to \overline{\partial}\phi$ in $L_\gamma^{n,1}(K)$ as $k \to \infty$. What we have to show is that
$$\overline{\partial}\mu_k \wedge \phi \to 0 \tag{52}$$
in $L_\gamma^{n,1}(K)$ as $k \to \infty$. By definition,
$$|\overline{\partial}\mu_k|_\gamma^2 \leq \frac{|\rho_k'(\log(-\log(r(\pi^* F))))|^2}{r^2(\pi^* F)\log^2(r(\pi^* F))}|r'|^2|\overline{\partial}\pi^* F|_\gamma^2 \tag{53}$$
$$\lesssim \frac{\chi_k(\pi^* F)}{(\pi^* F)^2 \log^2(\pi^* F)}, \tag{54}$$
where χ_k is the characteristic function of $[e^{-e^{k+1}}, e^{-e^k}]$ for $|\pi^*\overline{\partial}F|_\gamma = |\overline{\partial}F|_h \lesssim 1$ and μ_k is constant outside $[e^{-e^{k+1}}, e^{-e^k}]$.

We may assume that U is an open set in $\mathbb{C}^n$ and that E is just the normal crossing $\{z_1 \cdots z_d = 0\}$. The unreduced exceptional divisor $Z = \pi^{-1}(\{0\})$ is given as the common zero set of the holomorphic functions $\{\pi^* w_1, ..., \pi^* w_L\}$. Let Z have the order $k_j \geq 1$ on $\{z_j = 0\}$, i.e. assume that Z is given by the holomorphic function $f = z_1^{k_1} \cdots z_d^{k_d}$. It follows that

$$\pi^* F \sim |z_1^{k_1} \cdots z_d^{k_d}|,$$

and (54) yields

$$|\overline{\partial}\mu_k|_\gamma \lesssim |z_1|^{-k_1} \cdots |z_d|^{-k_d} |\log|^{-1}(|z_1 \cdots z_d|). \tag{55}$$

The assumption $\phi \in \Gamma(U, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$ implies that

$$z_1^{1-k_1} \cdots z_d^{1-k_d} \phi \in \Gamma(U, \mathcal{K}_M)$$

which does almost compensate the right hand-side of (55). We have to take care of the additional factor $\lambda := |z_1 \cdots z_d| |\log||z_1 \cdots z_d|$, but this is easy because $z_1^{1-k_1} \cdots z_d^{1-k_d} \phi$ has smooth coefficients and λ^{-1} is locally square-integrable in $\mathbb{C}^n$. Thus

$$\frac{\phi}{z_1^{k_1} \cdots z_d^{k_d} \log|z_1 \cdots z_d|} \in L_\gamma^{n,0}(K) = L_\sigma^{n,0}(K)$$

on the compact subset $K \subset U$. Combining this with (55), we see that $\overline{\partial}\mu_k \wedge \phi$ is uniformly bounded in $L_\gamma^{n,1}(K)$, and so

$$\overline{\partial}\mu_k \wedge \phi \to 0 \quad \text{in } L_\gamma^{n,1}(K)$$

because the domain of integration vanishes as $k \to \infty$. Hence we have just proved:

Lemma 6.1.

$$\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z) \subset \mathcal{F}_{\gamma, E}^{n,0}.$$

Let us now show that the $\overline{\partial}_s$-complex (51) is exact at $\pi_* \mathcal{F}_{\gamma, E}^{n,0}$. It is enough to consider a point $p \in E$ and a neighborhood U of p such that U is an open set in $\mathbb{C}^n$, that E is the normal crossing $\{z_1 \cdots z_d = 0\}$, and that $p = 0$.

Let us investigate the behavior of $(0,1)$-forms under the resolution $\pi : M \to X$ at the isolated singularity $\pi(p)$. We assume again that a neighborhood of $\pi(p)$ is embedded holomorphically into $U \subset\subset \mathbb{C}^L$, $L \gg n$, such that $\pi(p) = 0$, and that $\gamma = \pi^* h$ where h is the Euclidean metric in $\mathbb{C}^L$. Let $w_1, ..., w_L$ be the Cartesian coordinates of $\mathbb{C}^L$. We are interested in the behavior of the forms $\eta_\mu := \pi^* dw_\mu$ at the exceptional set. Let $dz_N := dz_1 \wedge \cdots \wedge dz_n$. By Lemma 2.1, a form α is in $L_\gamma^{n,q}(U)$ exactly if it can be written in multi-index notation as

$$\alpha = \sum_{|K|=q} \alpha_K dz_N \wedge \eta_K \tag{56}$$

with coefficients $\alpha_K \in L_\sigma^{0,0}(U)$. This representation is not unique.

Let Z have the order $k_j \geq 1$ on $\{z_j = 0\}$, i.e. assume again that Z is given by $f = z_1^{k_1} \cdots z_d^{k_d}$. Then each $\pi^* w_\mu$ has a factorization

$$\pi^* w_\mu = f g_\mu = z_1^{k_1} \cdots z_d^{k_d} \cdot g_\mu,$$

where g_μ is a holomorphic function on U. So,

$$\eta_\mu = \pi^* dw_\mu = d\pi^* w_\mu = \left(z_1^{k_1-1} \cdots z_d^{k_d-1} \right) \cdot \beta_\mu,$$

where the β_μ are $(0,1)$-forms that are bounded with respect to the non-singular metric σ. This means that $\eta_\mu = \pi^* dw_\mu$ vanishes at least to the order of $Z - |Z|$ along the exceptional set E (with respect to the metric σ).

So, (56) implies that a form α is in $L_\gamma^{n,1}(U)$ exactly if it can be written in multi-index notation as

$$\alpha = z_1^{k_1-1} \cdots z_d^{k_d-1} \sum_{j=1}^{L} \alpha_j dz_N \wedge \beta_j \tag{57}$$

with coefficients $\alpha_j \in L_\sigma^{0,0}(U)$.

Let $\phi \in \Gamma(U, \mathcal{F}_{\gamma,E}^{n,0})$ such that $\overline{\partial}_{s,loc}\phi = 0$. It is clear that $\phi \in \Gamma(U, \mathcal{K}_M)$, but we can show now that ϕ is actually vanishing to the order of $|Z| - Z$ along the exceptional set by use of Cauchy's integral formula in one complex variable. By the extension Lemma 2.7, it is enough to show that $\phi \in L_{\sigma,loc}^{n,0}(U, L_{|Z|-Z})$.

By definition, there exists a sequence of smooth forms with support away from the exceptional set E, $\{\phi_j\} \subset A_{c,E}^{n,0}(U)$, such that

$$\phi_j \quad \rightarrow \quad \phi \ \text{ in } L_\gamma^{n,0}(K),$$
$$\overline{\partial}\phi_j \quad \rightarrow \quad \overline{\partial}_w\phi \ \text{ in } L_\gamma^{n,1}(K),$$

on compact subsets $K \subset U$ as $j \to \infty$. Since we treat a local question at $0 \in \mathbb{C}^n$, it does no harm to work on a suitable neighborhood of the origin and to cut-off ϕ and the ϕ_j by a real-valued smooth function $\chi \in C_{cpt}^\infty(\mathbb{C})$ satisfying $\chi(z_1) = 1$ for $|z_1| \leq \epsilon$, $\chi(z_1) = 0$ for $|z_1| \geq 2\epsilon$, and $|\chi'| \leq 2\epsilon^{-1}$ for a fixed $\epsilon > 0$ small enough. So, replace $\phi(z)$ by $\phi(z)\chi(z_1)$ and $\phi_j(z)$ by $\phi_j(z)\chi(z_1)$. Note that the new $(n,0)$-forms are not holomorphic any more.

Because the ϕ_j have compact support away from E, we have the representation

$$\phi_j(z) = \frac{z_1^{k_1-1}}{2\pi i} \int_\mathbb{C} \frac{\partial \phi_j}{\partial \zeta_1}(\zeta_1, z_2, ..., z_n) \frac{d\zeta_1 \wedge d\zeta_1}{\zeta_1^{k_1-1}(\zeta_1 - z_1)}, \tag{58}$$

omitting dz_N in the notation for simplicity. But $\overline{\partial}\phi_j \to \overline{\partial}_w\phi$ in $L_\gamma^{n,1}(K)$ and the representation (57) imply that

$$\zeta_1^{-k_1+1}\overline{\partial}\phi_j \to \zeta_1^{-k_1+1}\overline{\partial}\phi$$

in the L^2-sense with respect to the non-singular metric σ. But the Cauchy formula (58) is bounded as an operator $L^2 \to L^2$.

Hence, the formula (58) converges to

$$\phi(z) = \frac{z_1^{k_1-1}}{2\pi i} \int_{\mathbb{C}} \frac{\partial \phi}{\partial \zeta_1}(\zeta_1, z_2, ..., z_n) \frac{d\zeta_1 \wedge d\zeta_1}{\zeta_1^{k_1-1}(\zeta_1 - z_1)}$$

in $L_\sigma^{n,0}$, and the integral on the right-hand side is itself in $L_\sigma^{n,0}$. Thus, we obtain $z_1^{1-k_1}\phi \in L_\sigma^{n,0}$. Similarly, we have $z_j^{1-k_j}\phi \in L_\sigma^{n,0}$ for $j = 2, ..., d$. But ϕ is an ordinary (smooth) holomorphic $(n, 0)$-form by the extension Lemma 2.7. So, it follows that $\phi \in L_{\sigma,loc}^{n,0}(U, L_{|Z|-Z})$. Thus we have proved:

Lemma 6.2. *For an open set $U \subset M$:*

$$\Gamma(U, \mathcal{F}_{\gamma,E}^{n,0}) \cap \ker \overline{\partial}_{s,loc} = \Gamma(U, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)).$$

Lemma 6.1 and Lemma 6.2 yield exactness of the direct image complex

$$0 \to \pi_* (\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \hookrightarrow \pi_* \mathcal{F}_{\gamma,E}^{n,0} \xrightarrow{\pi_* \overline{\partial}_s} \pi_* \mathcal{F}_{\gamma,E}^{n,1}.$$

It remains to show that

$$\pi_* \mathcal{F}_{\gamma,E}^{n,q-1} \xrightarrow{\pi_* \overline{\partial}_s} \pi_* \mathcal{F}_{\gamma,E}^{n,q} \xrightarrow{\pi_* \overline{\partial}_s} \pi_* \mathcal{F}_{\gamma,E}^{n,q+1} \tag{59}$$

is exact if $q \geq 1$. But this follows directly from Lemma 5.4 when $q \leq n - 1$: It is enough to consider an isolated singularity $p \in \operatorname{Sing} X$. We can assume that X is an analytic set in $\mathbb{C}^L$ which carries the restriction of the Euclidean metric, and that $p = 0 \in \mathbb{C}^L$. Let U be a neighborhood of 0 in X, $U^* = U - \{0\}$, and

$$f \in \pi_* \mathcal{F}_{\gamma,E}^{n,q}(U) \ , \ \pi_* \overline{\partial}_s f = 0.$$

That means nothing else but

$$f \in L^{n,q}(X_c^*) \ , \ \overline{\partial}_{s,w} f = 0,$$

in the notation of the previous section if we choose $c > 0$ such that $X_c \subset U$. By Lemma 5.4, there exists

$$g \in L^{n,q-1}(X_c^*) \ , \ \overline{\partial}_{s,w} g = f.$$

If $V \subset\subset X_c$, then it is clear that

$$g|_V \in \pi_* \mathcal{F}_{\gamma,E}^{n,q-1}(V) \ , \ \pi_* \overline{\partial}_s(g|_V) = f,$$

and that shows exactness of (59) for $1 \leq q \leq n - 1$.

The case $q = n$ is covered by the following observation, which in fact treats the situation $q \geq 2$. It is well-known that the $\overline{\partial}_w$-equation can be solved at isolated singularities with a gain of regularity that is enough to ensure that the solution is in the domain of $\overline{\partial}_s$ by the cut-off procedure we have used before. Fornæss, Øvrelid and Vassiliadou showed:

Lemma 6.3. *Let X be a pure n-dimensional complex analytic set in $\mathbb{C}^L$ with an isolated singularity at the origin such that $\operatorname{Reg} X$ carries the restriction of the Euclidean metric. For $s > 0$, let $X_s^* := X \cap B_s(0) - \{0\}$.*

Let $p + q \geq n + 2$, $q > 0$. Given $f \in L^2_{p,q}(X_r^*)$, $\overline{\partial}_w f = 0$ in X_r^*, there exists u satisfying $\overline{\partial}_w u = f$ in X_r^* with

$$\int_{X_{r_0}^*} \|z\|^{-2} \left(-\log \|z\|^2\right)^{-2} |u|^2 dV \leq C(r_0) \int_{X_r^*} |f|^2 dV$$

where $0 < r_0 < r$ and $C(r_0)$ is a positive constant that depends on r_0. u can be approximated by a sequence of smooth forms u_k with compact support away from the origin such that $u_k \to u$, $\overline{\partial} u_k \to f$ in $L^2_{p,*}(X_{r_0}^*)$, thus $\overline{\partial}_s u = f$.

Proof. The first statement is exactly Theorem 1.2 in [FOV2]. The second statement $\overline{\partial}_s u = f$ follows as the last statement of Theorem 1.1 in [FOV2], or by the cut-off procedure of our Lemma 6.1. It should be remarked that Fornæss, Øvrelid and Vassiliadou use the same cut-off procedure for the last statement of their Theorem 1.1, also inspired by Pardon and Stern [PS1]. $\square$

That shows exactness of (59) if $q \geq 2$. Hence, $(\pi_* \mathcal{F}^{n,*}_{\gamma,E}, \pi_* \overline{\partial}_s)$ is a fine resolution of $\pi_* (\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$, and that finishes the proof of Theorem 1.6.

We conclude this section by comparing $\overline{\partial}_w$- and $\overline{\partial}_s$-Dolbeault cohomology:

Theorem 6.4. *Let X be a Hermitian compact complex space of pure dimension with isolated singularities, $\pi : M \to X$ a resolution of singularities with only normal crossings, and $0 \leq q \leq n = \dim X$. Let Z be the unreduced exceptional divisor $Z := \pi^{-1}(\operatorname{Sing} X)$, $L \to M$ the holomorphic line bundle associated to $Z - |Z|$ such that $\mathcal{O}(L) \cong \mathcal{O}(Z - |Z|)$, and denote by $\pi_* L$ the push-forward to $X - \operatorname{Sing} X$ of the restriction of L to $M - |Z|$.*

If L is locally semi-negative with respect to X, then

$$H^{0,q}_w(X - \operatorname{Sing} X) \cong H^q(M, \mathcal{O}(Z - |Z|)) \cong H^{0,q}_s(X - \operatorname{Sing} X, \pi_* L). \tag{60}$$

Proof. The first isomorphism is given by Theorem 1.6, the second comes from Theorem 4.1. $\square$

This means that the $\overline{\partial}_w$-cohomology correspondents to the $\overline{\partial}_s$-cohomology for forms with poles determined by $Z - |Z|$. Note that the $\overline{\partial}_w$-cohomology is isomorphic to the usual $\overline{\partial}_s$-cohomology if $H^q(M, \mathcal{O}(Z - |Z|) \cong H^q(M, \mathcal{O}))$, which might well occur, especially if $q < n - 1$ (or trivially if $Z = |Z|$). See also [PS2] and [GL], [BL] for this topic.

7. About the non-compact case

It is worthwhile to remark that our considerations have been mainly of local nature (with respect to the base space X), and that compactness of X has been used basically to the extend that the domain where we consider the $\overline{\partial}$-equation has no boundary. So, if we consider a domain with a sufficiently nice boundary (smooth, strongly pseudoconvex) which does not intersect $\operatorname{Sing} X$, then our methods apply to this situation.

In this context, we need the following observation: if $D \subset\subset M$ is a smoothly bounded, strongly pseudoconvex domain in a complex manifold M, then there are natural isomorphisms $H_w^{p,q}(D) \cong H^{p,q}(D)$ and $H_s^{p,q}(D) \cong H_{cpt}^{p,q}(D)$ (see [H3, H4] or [FK]). In the first case, this means that the L^2-cohomology and the L_{loc}^2-cohomology coincide. In the second case, we see that it does not make a difference if we compute the cohomology with respect to compactly supported forms or with respect to forms just vanishing on the boundary of the domain (resp. forms that can be approximated in the L^2-category by compactly supported forms). Taking this into account, our methods yield the following statement:

Theorem 7.1. *Let (X, h) be a Hermitian complex space of pure dimension n, and $0 \le q \le n$. Let $D \subset\subset X$ be a domain with smooth, strongly pseudoconvex boundary such that $bD \cap \operatorname{Sing} X = \emptyset$. Then*
$$H_s^{0,q}(D - \operatorname{Sing} X) \cong H_{cpt}^q(\pi^{-1}(D), \mathcal{O})$$
for any resolution of singularities $\pi : M \to X$.

Assume in addition that X has only isolated singularities and that the exceptional divisor of the resolution π has only normal crossings. If $L_{Z-|Z|}$ is locally semi-negative with respect to X, then
$$H_w^{0,q}(D - \operatorname{Sing} X) \cong H^q(\pi^{-1}(D), \mathcal{O}(Z - |Z|)),$$
where $Z := \pi^{-1}(\operatorname{Sing} X)$ is the unreduced exceptional divisor.

Proof. We only sketch the main argument. As in the proof of Theorem 1.5, Lemma 3.1 and Lemma 3.5 imply that (with $E = |\pi^{-1}(\operatorname{Sing} X)|$, $\gamma = \pi^* h$)
$$H^q(\Gamma(D, \pi_* \mathcal{C}_{\sigma,E}^{n,*})) \cong H^q(D, \pi_* \mathcal{K}_M) \cong H^q(\Gamma(D, \pi_* \mathcal{C}_{\gamma,E}^{n,*})). \tag{61}$$
As indicated before, the fact that $\pi^{-1}(D)$ has smooth, strongly pseudoconvex boundary implies that there is no difference between L^2- and L_{loc}^2-cohomology:
$$H^q(\Gamma(D, \pi_* \mathcal{C}_{\sigma,E}^{n,*})) = H^q(\Gamma(\pi^{-1}(D), \mathcal{C}_{\sigma,E}^{n,*})) \cong H^{n,q}(\pi^{-1}(D)) \cong H_{cpt}^{n-q}(\pi^{-1}(D), \mathcal{O}).$$
On the other hand, the same is true for the L_γ^2-cohomology because the boundary of D does not intersect the singular set. Thus:
$$H^q(\Gamma(D, \pi_* \mathcal{C}_{\gamma,E}^{n,*})) = H^q(\Gamma(\pi^{-1}(D), \mathcal{C}_{\gamma,E}^{n,*})) \cong H_{\gamma,w}^{n,q}(\pi^{-1}(D)) \cong H_s^{0,n-q}(D - \operatorname{Sing} X).$$
That gives the first statement, the latter one can be proved similarly. $\square$

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